LEFT-DISTRIBUTIVE QUASIGROUPS

SHERMAN K. STEIN

A quasigroup $\otimes$ defined on the set S and satisfying the identity

$$(a \otimes b) \otimes (a \otimes c) = a \otimes (b \otimes c)$$

is left-distributive. Such quasigroups arise in various contexts, e.g., groups furnished with an automorphism leaving only the unit fixed [3] [5], totally symmetric quasigroups and Moufang loops [1], and conics [2; 4]. Also a left-distributive quasigroup has an orthogonal complement. Specifically, for $a \in S$ define the quasigroup $\circ$ on S by $x \circ y = x \otimes (a \otimes y)$. Then $\otimes$ is orthogonal to $\circ$. In [4] it is shown that there are no left-distributive quasigroups of order $4k+2$.

It will now be shown that left-distributive quasigroups are intimately connected with the binary operation of conjugation in a group.

THEOREM 1. *If $\otimes$ is a left-distributive quasigroup defined on the set S then there is a group defined on a set $T \supset S$ such that for all $x, y \in S$*

$$x \otimes y = xyx^{-1}.$$

PROOF. If $a \in S$ let left-translation $L_a: S \to S$ be defined by $L_a(x) = a \otimes x$. Then $\otimes$ is left-distributive if and only if

$$L_{a \otimes b} L_a = L_a L_b \qquad a, b \in S$$

where product is left composition of function. Since L_a is a bijection the preceding equation can be written

$$L_{a \otimes b} = L_a L_b (L_a)^{-1}.$$

Now let $\circ$ be the group generated by these left-translations and T' be the set on which $\circ$ is defined. Define the injection $i: S \to T'$ by $i(a) = L_a$. Set $T = (T' - i(S)) \cup S$. This S and T satisfy the stipulations of the theorem.

In the converse direction we have the easily verified

THEOREM 2. *Let $\circ$ be a group defined on a set T. Let $S \subset T$ satisfy the condition that for all $a, b, c \in S$, $b \neq c$,*

(1) *b and c are in distinct left cosets of the normalizer of a,*

(2) *$aba^{-1} \in S$.*

Then the operation $\otimes$, defined on S by

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$$x \otimes y = xyx^{-1}$$

is a left-distributive quasigroup on S.

This suggests

THEOREM 3. *If* $\circ$ *is a group defined on the set* T *then the operation* $\otimes$, *defined on* T *by*

$$x \otimes y = xyx^{-1}$$

is a left-distributive groupoid. Moreover for $a, b \in T$ *there is a unique* $x \in T$ *with* $a \otimes x = b$.

It might be pointed out that if $a \in T$ then the function $C_a: T \rightarrow T$, defined by $C_a(x) = axa^{-1}$, is an automorphism of $\otimes$, a consequence of the fact that C_a is an automorphism of $\circ$.

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UNIVERSITY OF CALIFORNIA, DAVIS