A NOTE ON EXTENDING SEMICHARACTERS ON SEMIGROUPS

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The purpose of this note is to give a necessary and sufficient condition for a semicharacter on a subsemigroup of a commutative semigroup G to be extendable to a semicharacter on G. A bounded complex function χ on a semigroup G is called a *semicharacter* of G if $\chi(x) \neq 0$ for some $x \in G$ and $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in G$. Semicharacters were introduced by Hewitt and Zuckerman [**1**] and in a slightly different form by Š. Schwarz, [**4**] and [**5**]. See also [**2**]. If χ is a semicharacter on G, then $|\chi(x)| \leqq 1$ for all $x \in G$. We will write $a|b$ if there exists an $x \in G$ such that $ax = b$.

We note that our theorem generalizes [**6**, Lemma 1, p. 94], and is related to [**3**, Lemma 1, p. 364]. Some remarks on the impossibility of generalizing our theorem appear at the end of the paper.

THEOREM. *Let G be a commutative semigroup and let $S \subseteq G$ be a subsemigroup. Then if χ is a semicharacter on S, the condition*

$$\text{(A)} \qquad a|b \quad \textit{and} \quad a, b \in S \quad \textit{imply} \quad |\chi(a)| \geqq |\chi(b)|,$$

is necessary and sufficient for χ to be extendable to a semicharacter on G.

PROOF. The necessity of (A) is clear. In proving the sufficiency, we first show that we may suppose that G has a unit. For suppose that the theorem is known for semigroups with unit. Then if G is a semigroup without unit, we adjoint a unit in the manner of [**1**, Theorem 2.3] and then apply the known result. The restriction to G of the semicharacter so obtained is the desired semicharacter. In the remainder of the proof it will be supposed that G has a unit e.

Let $\mathfrak{A}$ consist of all pairs $(\tilde{\chi}, T)$ such that

(1) $S \subseteq T$ and T is a subsemigroup of G,

(2) $\tilde{\chi}$ is a semicharacter on T,

(3) $\tilde{\chi}$ satisfies (A) on T, and

(4) $\tilde{\chi}$ extends χ.

We say that $(\tilde{\chi}_1, T_1) \leqq (\tilde{\chi}_2, T_2)$ if $T_1 \subseteq T_2$ and $\tilde{\chi}_2$ extends $\tilde{\chi}_1$. By Zorn's lemma $\mathfrak{A}$ has a maximal element (ψ, T_0). Clearly $e \in T_0$ for otherwise ψ can be trivially extended to the subsemigroup $T_0 \cup \{e\}$. If $T_0 = G$, the proof is complete. Suppose then that $T_0 \subset G$. In each of the follow-

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ing cases we will choose an $x_0 \in G - T_0$ and then extend ψ to a semicharacter $\tilde{\psi}$ on the subsemigroup $T_1 = \{tx_0^k : t \in T_0,\ k \geqq 0\}$ such that $\tilde{\psi}$ satisfies (A) on T_1. Since $T_1 \supset T_0$, this will constitute a contradiction.

CASE 1. Suppose there exist a_0, $b_0 \in T_0$ and $x_0 \in G - T_0$ such that $a_0 x_0 = b_0$ and $\psi(b_0) \neq 0$.

By (A) applied to T_0, $\psi(a_0) \neq 0$. Let $\tilde{\psi}(tx_0^k) = \psi(t)\psi(b_0)^k/\psi(a_0)^k$ for $tx_0^k \in T_1$. Suppose that $tx_0^k = ux_0^l$, where t, $u \in T_0$. Then $a_0^{k+l}tx_0^k = a_0^{k+l}ux_0^l$ and so $b_0^k a_0^l t = b_0^l a_0^k u$. Hence

$$\tilde{\psi}(tx_0^k) = \frac{\psi(t)\psi(b_0)^k}{\psi(a_0)^k} = \frac{\psi(u)\psi(b_0)^l}{\psi(a_0)^l} = \tilde{\psi}(ux_0^l).$$

Thus $\tilde{\psi}$ is well-defined. Since $a_0 | b_0$, we have $|\psi(a_0)| \geqq |\psi(b_0)|$. Hence $|\tilde{\psi}(tx_0^k)| \leqq |\psi(t)| \leqq 1$ for $tx_0^k \in T_1$; that is, $\tilde{\psi}$ is bounded. It is easy to show that $\tilde{\psi}$ extends ψ and is multiplicative.

We now show that $\tilde{\psi}$ satisfies (A) on T_1. Suppose that $tx_0^k | ux_0^l$, t, $u \in T_0$. Then $tx_0^k r = ux_0^l$ for some $r \in G$. Hence $a_0^{k+l}tx_0^k r = a_0^{k+l}ux_0^l$ or $b_0^k a_0^l tr = b_0^l a_0^k u$ so that $|\psi(b_0)^l\psi(a_0)^k\psi(u)| \leqq |\psi(b_0)^k\psi(a_0)^l\psi(t)|$. It follows that

$$|\tilde{\psi}(ux_0^l)| = \left|\frac{\psi(u)\psi(b_0)^l}{\psi(a_0)^l}\right| = \left|\frac{\psi(u)\psi(b_0)^l\psi(a_0)^k}{\psi(a_0)^{k+l}}\right| \leqq \left|\frac{\psi(b_0)^k\psi(a_0)^l\psi(t)}{\psi(a_0)^{k+l}}\right|$$
$$= \left|\frac{\psi(b_0)^k\psi(t)}{\psi(a_0)^k}\right| = |\tilde{\psi}(tx_0^k)|.$$

CASE 2. Suppose that Case 1 fails but that for some $x_0 \in G - T_0$ and some $k_1 \geqq 2$, we have $x_0^{k_1} \in T_0$ and $\psi(x_0^{k_1}) \neq 0$.

Let k_0 be the least positive integer such that $x_0^{k_0} \in T_0$. Since $x_0^{k_0} | x_0^{k_1}$, we have $\psi(x_0^{k_0}) \neq 0$. The denial of Case 1 is

(B) $\quad ax = b,\ a,\ b \in T_0, \quad \text{and} \quad x \in G - T_0 \quad \text{imply} \quad \psi(b) = 0.$

Let α be any k_0th root of $\psi(x_0^{k_0})$; that is, any complex number such that $\alpha^{k_0} = \psi(x_0^{k_0})$. Note that $T_1 = \{tx_0^k : t \in T_0,\ 0 \leqq k < k_0\}$. Let $\tilde{\psi}(tx_0^k) = \psi(t)\alpha^k$ for $tx_0^k \in T_1$, $0 \leqq k < k_0$.

To see that $\tilde{\psi}$ is well-defined, we first suppose that $tx_0^k = ux_0^k$ where t, $u \in T_0$, $0 \leqq k < k_0$. Then $tx_0^{k_0} = ux_0^{k_0}$. Hence $\psi(t)\psi(x_0^{k_0}) = \psi(u)\psi(x_0^{k_0})$ so that $\psi(t) = \psi(u)$. Consequently, $\tilde{\psi}(tx_0^k) = \tilde{\psi}(ux_0^k)$. Suppose now that $tx_0^k = ux_0^l$ where t, $u \in T_0$, $0 \leqq k < l < k_0$. Then $tx_0^{k+k_0-l} = ux_0^{k_0}$. By the minimality of k_0, $x_0^{k+k_0-l} \notin T_0$. Thus by (B), $\psi(ux_0^{k_0}) = 0$ and hence $\psi(u) = 0$. We also have $tx_0^{k_0} = ux_0^{k_0}x_0^{l-k}$. By the minimality of k_0, $x_0^{l-k} \notin T_0$ and hence by (B), $\psi(tx_0^{k_0}) = 0$. Thus $\psi(t) = 0$. Hence $\tilde{\psi}(tx_0^k)$

$=\tilde{\psi}(ux_0^l)=0$. It is routine to verify that $\tilde{\psi}$ is bounded, extends ψ, and is multiplicative.

We now check that $\tilde{\psi}$ satisfies (A) on T_1. Suppose that $tx_0^k | ux_0^l$, $t, u \in T_0$, $0 \leqq k < k_0$, $0 \leqq l < k_0$. Then for some $r \in G$, $tx_0^k r = ux_0^l$. Hence $t^{k_0}x_0^{k_0k}r^{k_0} = u^{k_0}x_0^{k_0l}$ so that $|\psi(u)^{k_0}\psi(x_0^{k_0})^l| \leqq |\psi(t)^{k_0}\psi(x_0^{k_0})^k|$ or $|\psi(u)^{k_0}\alpha^{k_0l}| \leqq |\psi(t)^{k_0}\alpha^{k_0k}|$. Then $|\psi(u)\alpha^l| \leqq |\psi(t)\alpha^k|$, i.e. $|\tilde{\psi}(ux_0^l)| \leqq |\tilde{\psi}(tx_0^k)|$.

Case 3. Suppose that Cases 1 and 2 fail but that for some

$$x_0 \in G - T_0 \text{ and } y_0 \in G,$$

we have $x_0y_0 \in T_0$ and $\psi(x_0y_0) \neq 0$.

The denials of Cases 1 and 2 are (B) and

(C) $\quad x \in G - T_0,\ k \geqq 2, \quad \text{and} \quad x^k \in T_0$ imply $\psi(x^k) = 0$.

Let $\alpha = \sup\left\{|\psi(x_0^k y)|^{1/k}\colon k \geqq 1,\ y \in G,\ x_0^k y \in T_0\right\}$; clearly $0 < \alpha \leqq 1$. Let $\tilde{\psi}(tx_0^k) = \psi(t)\alpha^k$ for $tx_0^k \in T_1$.

To see that $\tilde{\psi}$ is well-defined, we first suppose that $tx_0^k = ux_0^k$, $t, u \in T_0$, $k \geqq 0$. Then $tx_0^k y_0^k = ux_0^k y_0^k$ and hence $\psi(t)\psi(x_0y_0)^k = \psi(u)\psi(x_0y_0)^k$. Thus $\psi(t) = \psi(u)$ and, consequently, $\tilde{\psi}(tx_0^k) = \tilde{\psi}(ux_0^k)$. Suppose now that $tx_0^k = ux_0^l$, $t, u \in T_0$, $0 \leqq k < l$. Then $tx_0^k y_0^k = ux_0^k x_0^{l-k} y_0^k$. If $x_0^{l-k} \in T_0$, then by (C), $\psi(x_0^{l-k}) = 0$ and hence $\psi(t)\psi(x_0y_0)^k = 0$. If $x_0^{l-k} \notin T_0$, then by (B), $\psi(t)\psi(x_0y_0)^k = 0$. Hence in either case, $\psi(t) = 0$. Since $tx_0^k y_0^l = ux_0^l y_0^l$, we have $|\psi(u)\psi(x_0y_0)^l| \leqq |\psi(t)| = 0$. Hence $\psi(u) = 0$. Thus $\tilde{\psi}(tx_0^k) = \tilde{\psi}(ux_0^l) = 0$. It is easy to verify that $\tilde{\psi}$ is bounded, extends ψ, and is multiplicative.

We now show that $\tilde{\psi}$ satisfies (A) on T_1. Suppose that $tx_0^k | ux_0^l$, $t, u \in T_0$. Then $tx_0^k r = ux_0^l$ for some $r \in G$. Suppose that $k = l$. Then $tx_0^k y_0^k r = ux_0^k y_0^k$ and hence $|\psi(u)\psi(x_0y_0)^k| \leqq |\psi(t)\psi(x_0y_0)^k|$ so that $|\psi(u)| \leqq |\psi(t)|$. Therefore $|\tilde{\psi}(ux_0^k)| \leqq |\tilde{\psi}(tx_0^k)|$.

Suppose that $k > l$. It follows that $tx_0^{k-l}r(x_0y_0)^l = u(x_0y_0)^l$. If $x_0^{k-l}r \notin T_0$, then by (B), $\psi(u)\psi(x_0y_0)^l = 0$. Thus $\psi(u) = 0$ and hence $|\tilde{\psi}(ux_0^l)| = 0 \leqq |\tilde{\psi}(tx_0^k)|$. Suppose that $x_0^{k-l}r \in T_0$. Then $|\psi(x_0^{k-l}r)|^{1/k-l} \leqq \alpha$ or $|\psi(x_0^{k-l}r)| \leqq \alpha^{k-l}$. We then have

$$\begin{aligned} |\tilde{\psi}(ux_0^l)\psi(x_0y_0)^l| &= |\psi(u)\alpha^l\psi(x_0y_0)^l| = |\psi(t)\psi(x_0^{k-l}r)\psi(x_0y_0)^l\alpha^l| \\ &\leqq |\psi(t)\alpha^{k-l}\alpha^l\psi(x_0y_0)^l| = |\tilde{\psi}(tx_0^k)\psi(x_0y_0)^l|, \end{aligned}$$

from which it follows that $|\tilde{\psi}(ux_0^l)| \leqq |\tilde{\psi}(tx_0^k)|$.

Finally, suppose that $k < l$. Assume that $|\tilde{\psi}(tx_0^k)| < |\tilde{\psi}(ux_0^l)|$. Then $|\psi(t)\alpha^k| < |\psi(u)\alpha^l|$ or $|\psi(t)|^{1/l-k}/|\psi(u)|^{1/l-k} < \alpha$. Choose $m \geqq 1$ and $y \in G$ such that $x_0^m y \in T_0$ and $|\psi(x_0^m y)|^{1/m} > |\psi(t)|^{1/l-k}/|\psi(u)|^{1/l-k}$. It follows that $|\psi(x_0^m y)^{l-k}\psi(u)^m| > |\psi(t)^m|$ or

$$(1) \qquad |\psi(x_0^m y)^l \psi(u)^m| > |\psi(t)^m \psi(x_0^m y)^k|.$$

Now $tx_0^k r = ux_0^l$ implies that $t^m x_0^{km} r^m y^k y^{l-k} = u^m x_0^{lm} y^l$ so that $t^m(x_0^m y)^k | u^m(x_0^m y)^l$. Now applying (A) to $t^m(x_0^m y)^k$, $u^m(x_0^m y)^l \in T_0$, we get

$$|\psi(u)^m \psi(x_0^m y)^l| \leqq |\psi(t)^m \psi(x_0^m y)^k|$$

which contradicts (1). Hence $|\tilde{\psi}(ux_0^l)| \leqq |\tilde{\psi}(tx_0^k)|$.

CASE 4. Suppose that Cases 1, 2, and 3 fail.

The denial of Cases 1, 2, and 3 is

(D) $\quad x \in G - T_0,\ y \in G,$ and $xy \in T_0$ imply $\psi(xy) = 0.$

Let $x_0 \in G - T_0$ be arbitrary and fixed and let

$$\tilde{\psi}(tx_0^k) = \begin{cases} \psi(t) & \text{for } k = 0 \\ 0 & \text{for } k \geqq 1 \end{cases} \quad \text{for } tx_0^k \in T_1.$$

We first check that $\tilde{\psi}$ is well-defined. Suppose that $tx_0^k = ux_0^l$, $t, u \in T_0$, $0 \leqq k \leqq l$. If $k = l = 0$, then clearly $\tilde{\psi}(u) = \tilde{\psi}(t)$. If $k > 0$, then $\tilde{\psi}(tx_0^k) = \tilde{\psi}(ux_0^l) = 0$. If $k = 0$ and $l > 0$, then $t = ux_0^l = ux_0^{l-1}x_0 \in T_0$. By (D), $\psi(t) = 0$ and hence $\tilde{\psi}(t) = \tilde{\psi}(ux_0^l) = 0$. It is routine to check that $\tilde{\psi}$ is bounded, extends ψ, and is multiplicative.

We show finally that $\tilde{\psi}$ satisfies (A) on T_1. Suppose that $tx_0^k | ux_0^l$, $t, u \in T_0$. Clearly $|\tilde{\psi}(ux_0^l)| = 0 \leqq |\tilde{\psi}(tx_0^k)|$ unless $l = 0$, so we may suppose that $tx_0^k | u$. Clearly $|\psi(u)| \leqq |\psi(t)|$ if $k = 0$. Suppose that $k > 0$. Then $tx_0^k r = u$ for some $r \in G$. Since $(trx_0^{k-1})x_0 \in T_0$, we have by (D) that $\psi(u) = \psi(trx_0^k) = 0$. Thus $|\tilde{\psi}(ux_0^l)| = 0 \leqq |\tilde{\psi}(tx_0^k)|$.

This completes the proof.

REMARKS. Our theorem may be stated as follows, where H is the complex unit disk.

Let G be a commutative semigroup and let $S \subseteq G$ be a subsemigroup. Then if f is a homomorphism of S into H, the condition

(A) $\quad a | b$ *and* $a, b \in S$ *imply* $f(a) | f(b)$,

is necessary and sufficient for f to be extendable to a homomorphism of G into H.

A similar result is clearly valid if H is $[0, 1]$, $[0, \infty[$, or the entire complex plane. It is evident that the statement is not valid if H is a semigroup that does not have a zero or identity or is not divisible. The proof of Case 3 together with the following example shows that some sort of completeness property is needed for H in addition to divisibility and the existence of a zero and identity. Let $G = [0, 1]$,

$S=H=\{a\in[0, 1]: a$ is algebraic$\}$, and f be the identity map. It can be shown that f cannot be extended to G. Our theorem thus cannot be extended to the case in which H is a divisible commutative semigroup with zero and identity.

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