TAME TRIODS IN 3-SPACE

P. H. DOYLE[1]

A triod is a homeomorphic image of the set consisting of three linear intervals which are disjoint except for a single point which is an end point of each interval. The images of the intervals are called branches of the triod and their intersection the branch point. In [3] the author presented an example of a wild triod in euclidean 3-space, E^3, which has an open 3-cell complement in compactified E^3 while every arc of the triod is tame. The present paper gives two independent conditions that a triod in E^3 be tame.

THEOREM 1. *Let T be a triod in E^3. In order that T be tame it is necessary and sufficient that* (i) *all arcs lying in T be tame and* (ii) *two branches of T lie in the interior of a disk D which intersects the remaining branch only at the branch point.*

The necessity of the conditions (i) and (ii) is clear. That (i) and (ii) are independent follows from the example of [3] and the Example 1.1 of [4] with an arc attached. Evidently the condition (ii) can be replaced by the a priori stronger condition that T lie on a disk or a 2-sphere. Before giving the sufficiency argument for Theorem 1 we shall prove

THEOREM 2. *Let J be an arc which lies on the boundary C of a tetrahedron in E^3 and p a point in the interior of J. If L is a tame arc with p as an end point, while $L \cap C = p$, and if either arc formed by L and a component of $J \backslash p$ is tame, then $L \cup J$ is a tame triod.*

PROOF. Since the arc J lies on the boundary of a tetrahedron it is tame. Evidently there is no loss of generality in assuming that J is a linear interval on a face of C. We assume that this is the case.

Let A be the component of $J \backslash p$ for which $L \cup A$ is a tame arc. In the set C we select a polygonal simple closed curve S which intersects J at p only, while the components of $J \backslash p$ lie in different components of $C \backslash S$; denote by D the closure of the component of $C \backslash S$ which contains A and let D' be the other. Then D is a polyhedral 2-cell in C.

A swelling process is described in Lemma 5.1 of [5]. By use of this procedure we obtain a 2-cell D'' with S as its boundary such that the interior of D'' lies in the component of $E^3 \backslash C$ which intersects L,

Received by the editors October 27, 1958.

[1] The work on this paper was supported in part by the National Science Foundation Grant G-2793.

$D''\cap L=p$, $E^3/(D'\cup D'')$ has A in one component and $L\backslash p$ in the other, and D'' is locally polyhedral mod p.

Since D'' is locally polyhedral mod p while D'' has a tame boundary, D'' is tame by Theorem 2 of [6] and Lemma 5.1 of [8]. Hence the 2-sphere $D'\cup D''$ is tame by Theorem 9.3 of [8]. The arc $L\cup A$ pierces [7] $D'\cup D''$ at p and so $L\cup A\cup(D'\cup D'')$ is tame by Theorem 1 of [7]. Because $J\backslash A$ lies in $D'\cup D''$ it follows that $J\cup L$ is tame.

PROOF OF THEOREM 1. Let T be as described in Theorem 1. The two branches of T which lie in D constitute an arc J. Since D may be selected so that J lies in its boundary we suppose D has this property. We first show that D can be selected a tame disk. If x^n is an n-cell, ∂x^n is its combinatorial boundary.

By Theorem 7 of [1] we can assume that D is locally polyhedral mod ∂D. Let h be a homeomorphism of E^3 onto itself which carries J onto a linear interval while $h(D)$ is locally polyhedral mod $h(\partial D)$. That h exists follows from Theorem 2 of [6]. Let P and Q be small disjoint triangles which are pierced by $h(J)$ at the points a and b of $h(J)$; a and b are assumed to be interior points of $h(J)$ which are separated in $h(J)$ by $h(p)$ while ∂P and ∂Q link $h(\partial D)$. By Lemma 5.1 of [5] P and Q can be replaced by disjoint disks P' and Q' such that $\partial P'$ and $\partial Q'$ link $h(\partial D)$, P' and Q' are locally polyhedral mod a and b respectively, while $P'\cap h(D)$ and $Q'\cap h(D)$ are arcs, A and B, respectively and $h(J)$ pierces P' and Q' at a and b respectively. By Theorem 2 of [7] $P'\cup h(J)$ and $Q'\cup h(J)$ are tame. It follows that P' and Q' are tame and consequently their subsets A and B are tame arcs.

Let the end points of A and B which lie in the interior of $h(D)$ be joined in an arc R in the interior of $h(D)$, where $R\cap A$ and $R\cap B$ are points. The arcs A, B, and R along with the component of $h(\partial D)/(a, b)$ which contains $h(p)$ form a tame simple closed curve M. This follows since R lies in the interior of $h(D)$ where $h(D)$ is locally polyhedral. Let N be the disk which M bounds in $h(D)$. By Theorem 2 of [6] and Lemma 5.1 of [8] N is tame. Since the points a and b can be selected closer to the end points of $h(J)$ and a tame disk N' containing N can be found, it follows that $N\cup h(J)$ is locally tame and hence tame by [2] or [8]. Evidently the tame disk N lies in another tame disk H which contains $h(J)$ entirely and which meets the remaining branch of $h(T)$ only at $h(p)$. It follows that if $h^{-1}(H)$ is selected for D, then D is tame.

Since the disk D in Theorem 1 can be chosen tame we suppose that it is. The proof will now be completed by showing that T has the same positional property as the triod $L\cup J$ in Theorem 2.

Let J be the two branches of T in D and L the other branch.

Since D is tame there is a homeomorphism h_1 of E^3 onto itself which carries D onto a polyhedral disk, $h_1(D)$. The swelling process of Lemma 5.1 in [5] is now applied to $h_1(D)$ so as to obtain a 3-cell C^3 such that C^3 is locally polyhedral mod $h_1(\partial D)$, $h_1(D)\subset\partial C^3$, and $C^3\cap h_1(L)=h_1(p)$. By Theorem 9.3 of [8] ∂C^3 is tame and so C^3 is tame. Hence there is a homeomorphism h_2 of E^3 onto itself such that $h_2[h_1(C^3)]$ is a tetrahedron. We note that $h_2h_1(J)$ lies in the boundary of a tetrahedron, while $h_2h_1(L)$ is a tame arc meeting $h_2h_1(\partial C^3)$ at $h_2h_1(p)$. Since $h_2h_1(L)$ along with either component of $h_2h_1(J)\setminus[h_2h_1(p)]$ is a tame arc the hypothesis of Theorem 2 is met by $h_2h_1(T)$ and $h_2h_1(\partial C^3)$. Hence $h_2h_1(T)$ is tame and so T is tame.

References

1. R. H. Bing, *Approximating surfaces with polyhedral ones*, Ann. of Math. vol. 65 (1957) pp. 456–483.

2. ———, *Locally tame sets are tame*, Ann. of Math. vol. 59 (1954) pp. 145–158.

3. P. H. Doyle, *A wild triod in 3-space*, Duke Math. J. vol. 26 (1959) pp. 263–268.

4. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. vol. 49 (1948) pp. 979–990.

5. O. G. Harrold, H. C. Griffith and E. E. Posey, *A characterization of tame curves in three-space*, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 12–34.

6. E. E. Moise, *Affine structures in 3-manifolds*, V. *The triangulation theorem and hauptvermutung*, Ann. of Math. vol. 56 (1952) pp. 96–114.

7. ———, *Affine structures in 3-manifolds*, VII. *Disks which are pierced by intervals*, Ann. of Math. vol. 58 (1953) pp. 403–408.

8. ———, *Affine structures in 3-manifolds*, VIII. *Invariance of the knot-types; local tame imbedding*, Ann. of Math. vol. 59 (1954) pp. 159–170.

Michigan State University and
University of Tennessee