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ONE DIMENSIONAL TOPOLOGICAL LATTICES

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1. By a *topological lattice* we mean a Hausdorff space L and a pair of continuous functions $\vee: L\times L\to L$ and $\wedge: L\times L\to L$ which satisfy the usual conditions stipulated for a lattice in Birkhoff [2, p. 18]. The purpose of this paper is to prove the

Main Theorem. *A locally compact, connected, one dimensional topological lattice is a chain.*

The lattice theoretic terminology used in this paper is consistent with Birkhoff [2]. The topological terms can be found in [5] or [8] with the following exceptions. If X and Y are sets, $X\backslash Y$ denotes the relative compliment of Y with respect to X. If A is a subset of a topological space then A^*, A^0 and $F(A)=A^*\backslash A^0$ denote the topological closure, interior and boundary of A. The symbol $\varnothing$ denotes the empty set.

We will agree that in usage of words common to topology and lattice theory, the topological meaning will take precedence. Thus to say that a subset A of a topological lattice is *closed* means $A=A^*$ and *not* $A\wedge A\subset A$ or $A\vee A\subset A$.

If L is a lattice and A is a subset of L we let

$$C(A) = (A \wedge L) \cap (A \vee L).$$

If $A=C(A)$ we say that A is a *convex* subset of L. It is clear that the set A is convex if, and only if, $x\vee(y\wedge L)\subset A$ whenever x and y are elements of A with $x\leqq y$. A topological lattice is *locally convex* if, and only if, whenever x is an element of an open set U there is an open convex set V with $x\in V\subset U$.

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We reserve the symbols **0** and **1** to denote the minimal and maximal elements of a lattice whenever they exist.

Examples of topological lattices may be found in [1] or [4]. The Euclidean plane with the usual topology and coordinatewise lattice operations is a simple but useful example of a topological lattice.

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2. L. Nachbin [6] showed that a compact partially ordered topological space is locally convex. In this section we modify his result and prove

THEOREM 1. *A locally compact, connected, topological lattice is locally convex.*

Some preliminary lemmas are in order before we proceed with the proof of Theorem 1.

LEMMA 1. *If L is a connected topological lattice and if A is a subset of L with $A\wedge L\subset A$ or $A\vee L\subset A$ then A is connected.*

PROOF. If $a\in L$ then the function $f(x)=a\wedge x$ is continuous and so $f(L)=a\wedge L$ is a connected set. Now if $A\wedge L\subset A$ and if x and y are elements of A then $x\wedge L\subset A$ and $y\wedge L\subset A$. Also $x\wedge L$ and $y\wedge L$ are connected sets with $(x\wedge L)\cap(y\wedge L)\neq\varnothing$ and so $(x\wedge L)\cup(y\wedge L)$ is a connected subset of A which contains both x and y. Therefore A is connected.

LEMMA 2. *If L is a topological lattice and if U is an open subset of L then $U\wedge L$ and $U\vee L$ are open.*

PROOF. We will show that $U\wedge L$ is open. Suppose $\{x_\alpha\}$ is a net in $L\backslash(U\wedge L)$ and $\{x_\alpha\}$ converges to some element y in L. If $y\in U\wedge L$ then for some $z\in U$, $y\wedge z=y$. Now $\vee$ is a continuous function and so the net $\{x_\alpha\vee z\}$ converges to $y\vee z=z$ and z is an element of U. Since U is open, there is an x_α such that $x_\alpha\vee z\in U$ and so $x_\alpha=(x_\alpha\vee z)\wedge x_\alpha\in U\wedge L$ which is a contradiction. Thus $y\in L\backslash(U\wedge L)$ and therefore $L\backslash(U\wedge L)$ is closed which implies that $U\vee L$ is open. A dual argument will show that $U\vee L$ is open

LEMMA 3. *If L is a topological lattice and if $A\subset L$ is compact then $A\wedge L$ and $A\vee L$ are closed.*

PROOF. We will show that $A\wedge L$ is closed. Suppose x is not an element of $A\wedge L$. Then $x\wedge A$ is a compact set which does not contain x, hence there are disjoint open sets U and V such that $x\in U$ and $x\wedge A\subset V$. Now $\wedge$ is continuous and A is compact, hence there is an open set W such that $x\in W$ and $W\wedge A\subset V$. Now if $y\in(W\cap U)\cap(A\wedge L)$ then for some $a\in A$, $y\wedge a=y$ and therefore $y=a\wedge y\in A\wedge W\subset V$ and $y\in W\cap U\subset U$ which contradicts the choice of U and V. Thus $W\cap U$ is an open subset of $L\backslash(A\wedge L)$ which contains x and so it follows that $A\wedge L$ is closed. Dually we get $A\vee L$ closed.

Combining Lemmas 2 and 3 we have

LEMMA 4. *If L is a topological lattice and if A is a subset of L, then*
(i) *$C(A)$ is closed whenever A is compact.*
(ii) *$C(A)$ is open whenever A is open.*

We now prove Theorem 1. Let us assume that a is an element of an open subset U of L. We must show there is an open set V with $a\in C(V)\subset U$. To this end, let us assume that this proposition is false, i.e. that $(L\backslash U)\cap C(V)\neq\varnothing$ for all open sets V which contain a. Now L is locally compact, hence there is an open set P such that $a\in P\subset P^*\subset U$ and P^* is compact. Let $\mathcal{W}$ denote the collection of all open sets $W\subset L$ such that $a\in W\subset W^*\subset P$. Since we have assumed that $C(W)\cap(L\backslash U)\neq\varnothing$ for all $W\in\mathcal{W}$, it follows that $C(W^*)\cap(L\backslash P)\neq\varnothing$ for all $W\in\mathcal{W}$. We will now show that $C(W^*)\cap F(P)\neq\varnothing$ for all $W\in\mathcal{W}$. If $W\in\mathcal{W}$ and if $z\in C(W^*)\cap(L\backslash P)$ then for some $x\in W^*$ we have $z=z\wedge x$ and therefore $x,\ z\in z\vee(x\wedge L)\subset C(W^*)$. By Lemma 1, $z\vee(x\wedge L)$ is connected and since $z\vee(x\wedge L)$ meets both P and $L\backslash P$, $(z\vee(x\wedge L))\cap F(P)\neq\varnothing$. But $z\vee(x\wedge L)\subset C(W^*)$ and so $C(W^*)\cap F(P)\neq\varnothing$. From the regularity of L it follows that the collection of closed sets $\{C(W^*)\cap F(P)\colon W\in\mathcal{W}\}$ has the finite intersection property. Therefore, since P^* is compact, we have $\bigcap\{C(W^*)\cap F(P)\colon W\in\mathcal{W}\}\neq\varnothing$. However, it is easily seen that $\bigcap\{C(W^*)\colon W\in\mathcal{W}\}=(a\wedge L)\cap(a\vee L)=\{a\}$ and so $\{a\}=\bigcap\{C(W^*)\cap F(P)\colon W\in\mathcal{W}\}\subset F(P)$ which is a contradiction. Hence the result is established.

3. In this section we prove the main theorem. We first prove a few preliminary results.

LEMMA 5. *If L is a topological lattice and if $a\in L$ then $(a\wedge L)\vee F(a\wedge L)=F(a\wedge L)$ and $(a\vee L)\wedge F(a\vee L)=F(a\vee L)$.*

PROOF. Since $(a\wedge L)\vee[L\backslash(a\wedge L)]\subset L\backslash(a\wedge L)$ and $\vee$ is continuous, it follows that $(a\wedge L)\vee[L\backslash(a\wedge L)]^*\subset[L\backslash(a\wedge L)]^*$. Also $(a\wedge L)\vee(a\wedge L)\subset(a\wedge L)$ and therefore $(a\wedge L)\vee F(a\wedge L)=(a\wedge L)\vee[(a\wedge L)\cap(L\backslash(a\wedge L))^*]\subset(a\wedge L)\cap[L\backslash(a\wedge L)]^*=F(a\wedge L)$. Since $\vee$ is an idempotent function and $F(a\wedge L)\subset a\wedge L$ we have $F(a\wedge L)\subset(a\wedge L)\vee F(a\wedge L)$ and so the result is established. A dual argument establishes that $(a\vee L)\wedge F(a\vee L)=F(a\vee L)$.

If L is a lattice and if $a \in L$, let

$$U(a) = L \backslash [(a \wedge L) \cup (a \vee L)].$$

Clearly $U(a)$ is the set of elements in L that are not comparable to a.

LEMMA 6. *If L is a connected topological lattice and if $a \in L$ then*
(i) *$F(a \wedge L)$ and $F(a \vee L)$ are connected.*
(ii) $a \wedge [L \backslash (a \wedge L)] \subset F(a \wedge L)$ and $a \vee [L \backslash (a \vee L)] \subset F(a \vee L)$.
(iii) *$a \in U(a)^*$ whenever $U(a)$ is not void.*

PROOF OF (i). Clearly $a \wedge L$ is a sublattice of L and by Lemma 1, $a \wedge L$ is connected. Now by Lemma 5, $F(a \wedge L) \vee (a \wedge L) = F(a \wedge L)$ and so by Lemma 1, it follows that $F(a \wedge L)$ is connected. A dual argument shows that $F(a \vee L)$ is connected.

PROOF OF (ii). If x is an element of $L \backslash (a \wedge L)$ then $x \wedge L$ contains both x and $a \wedge x$, hence meets both $a \wedge L$ and $L \backslash (a \wedge L)$. Since $x \wedge L$ is a connected set, it follows that $x \wedge L$ also meets $F(a \wedge L)$. Let $y \in (x \wedge L) \cap F(a \wedge L)$ then $x \wedge y = y$ and $a \wedge y = y$ and so $(a \wedge x) \wedge y = y$ which implies that $(a \wedge x) \vee y = a \wedge x$. Now by Lemma 5, we have $(a \wedge x) \vee y \in F(a \wedge L)$ and so $a \wedge x \in F(a \wedge L)$. Dually, we establish that $a \vee [L \backslash (a \vee L)] \subset F(a \vee L)$.

PROOF OF (iii). If b is an element of $U(a)$ then $a \wedge b$ is an element of $F(a \wedge L)$ and $a \wedge b \neq a$. Since $F(a \wedge L)$ is connected and contains points other than a, it follows that a is an accumulation point of $F(a \wedge L)$. Now if V is an open set containing a, then $V \cap (L \backslash (a \vee L)) \cap F(a \wedge L) = V \cap (F(a \wedge L) \backslash \{a\}) \neq \varnothing$. Therefore we have that $V \cap (L \backslash (a \wedge L)) \cap (L \backslash (a \vee L)) = V \cap U(a) \neq \varnothing$ and so $a \in U(a)^*$.

If X is a space and A a subset of X, we denote by $H^n(X, A)$ the n-dimensional Alexander-Kolomogoroff cohomology group of X modulo A with coefficients in some nontrivial abelian group with identity e. We note that $H^n(X, \varnothing) \equiv H^n(X)$. We recall that a locally compact Hausdorff space X has codimension at most n if, and only if, for each compact $A \subset X$ and each closed $B \subset A$ if $i: B \rightarrow A$ is the injection function then the induced homomorphism $i^*: H^n(B) \rightarrow H^n(A)$ is onto. For the essential properties of this dimension function, the reader is referred to [**3**].

LEMMA 7. *A compact connected topological lattice of codimension at most one is a chain.*

PROOF. Let L be a compact connected topological lattice of codimension at most one and suppose L contains two unrelated elements, a and b. By the Hausdorff Maximality Principle, there are maximal chains M and N which contain a and b respectively. Since

L is compact and connected, both M and N are closed, connected and contain $\mathbf{0}$ and $\mathbf{1}$ [9]. It is also easily seen that any proper subset of M(or N) which contains $\mathbf{0}$ and $\mathbf{1}$ is not connected.

We now consider the following Mayer-Vietoris exact sequence [8] (we denote the set $\{\mathbf{1}\}$ by $\mathbf{1}$)

$$H^0(M, \mathbf{1}) \times H^0(N, \mathbf{1}) \xrightarrow{J^*} H^0(M \cap N, \mathbf{1}) \xrightarrow{\Delta^*} H^1(M \cup N, \mathbf{1})$$
$$\xrightarrow{I^*} H^1(M, \mathbf{1}) \times H^1(N, \mathbf{1}).$$

It is proven in [7] that the groups of a compact connected topological lattice vanish in all dimensions greater than 0, thus we have $H^1(M, \mathbf{1}) = H^1(N, \mathbf{1}) = e$. Since M and N are connected, it follows that $H^0(M, \mathbf{1}) = H^0(N, \mathbf{1}) = e$ and therefore Δ^* is an isomorphism onto. Now $M \cap N$ is a proper subset of M which contains $\mathbf{0}$ and $\mathbf{1}$, therefore $M \cap N$ is not connected. Thus it follows that $H^0(M \cap N, \mathbf{1}) \neq e$ and so $H^1(M \cup N) \approx H^1(M \cup N, \mathbf{1}) \neq e$. Also, L is a compact connected topological lattice and so by [7], $H^1(L) = e$. However, the codimension of L is at most one and $M \cup N$ is a closed subset of L, therefore if i: $M \cup N \to L$ is the injection function, then i^*: $H^1(L) \to H^1(M \cup N)$ is onto which is a contradiction.

We note that Lemma 7 apparently requires the use of algebraic topology. At least the author knows no purely set theoretic proof of this lemma.

MAIN THEOREM. *A locally compact connected topological lattice of codimension at most one is a chain.*

PROOF. Suppose L is a locally compact connected topological lattice with codimension at most one and that a is an element of L with $U(a)$ not empty. By Theorem 1, L is locally convex and so there is an open convex set V containing a with V^* compact. Since $\vee$ and $\wedge$ are continuous and $a \vee a = a = a \wedge a$, there is an open set W containing a with $W \vee W \subset V$ and $W \wedge W \subset V$. By Lemma 6, we have $a \in U(a)^*$ and so $W \cap U(a) \neq \emptyset$. Let $b \in W \cap U(a)$ so that $a \wedge b \in W \wedge W \subset V$ and $a \vee b \in W \vee W \subset V$. Letting $B = (a \wedge b) \vee ((a \vee b) \wedge L)$ we have $B \subset V$ since V is convex. Also B is a closed subset of V^*, hence compact. Clearly B is a sublattice of L and, by Lemma 1, B is connected, therefore B is a compact connected sublattice of L. Now B is closed and the codimension of L is at most one, hence the codimension of B is at most one. Therefore by Lemma 7, B is a chain but this is contradiction since B contains both a and b.

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