ON THE ABSOLUTE HARMONIC SUMMABILITY OF A SERIES RELATED TO A FOURIER SERIES

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1.1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let the sequence $\{t_n\}$ be defined by

$$t_n = \frac{(n+1)^{-1}s_0 + n^{-1}s_1 + \cdots + 1\cdot s_n}{P_n},$$

$$\left(P_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right).$$

The series $\sum a_n$ is defined to be summable by harmonic means if the sequence $\{t_n\}$ tends to a limit as $n\to\infty$ [4]. If the series $\sum_n |t_n - t_{n-1}|$ is convergent, we say that the series is absolutely harmonic summable. It is known that the method of summability is absolutely regular and implies absolute Cesàro summability of every positive order [2].

1.2. Let $f(t)$ be a periodic function, with period 2π, and integrable (L) over $(-\pi, \pi)$. We assume without any loss of generality that the Fourier series of $f(t)$ is given by

$$\sum_1^\infty (a_n \cos nt + b_n \sin nt) = \sum_1^\infty A_n(t), \tag{1.21}$$

and that $\int_{-\pi}^{\pi} f(t)dt = 0$. We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}.$$

Mohanty [3] has considered the absolute Riesz summability of the series

$$\sum_{n=1}^\infty A_n(t)/\log(n+1). \tag{1.22}$$

2. In this paper we establish the following theorem:

THEOREM. *If $\phi(t)$ is of bounded variation in $(0, \pi)$ then the series (1.22) is absolutely summable by harmonic means.*

We require the following lemmas for the proof of our theorem:

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LEMMA 1 [5]. *Uniformly for* $0<t<\pi$

$$\left|\sum_{m}^{n} \frac{\sin \nu t}{\nu}\right| \leqq K$$

where m and n are any positive integers.

LEMMA 2 [1]. *If* $0<t<\pi$, *then*

$$\left|\sum_{k=0}^{m} \frac{\cos(k+1)t}{k+1}\right| = O\left(1+\log\frac{1}{t}\right).$$

With the help of Lemmas 1 and 2, we may easily deduce

LEMMA 3. *If* $0<t<\pi$, *then for all positive integers m and m'*

$$\sum_{k=m}^{m'} \frac{\sin(n-k)t}{k+1} = O\left(1+\log\frac{1}{t}\right).$$

LEMMA 4. *If* $P_n=1+1/2+\cdots+1/(n+1)$, *then*

$$\text{(i)} \qquad \sum_{k=0}^{[n/2]-2} \left|\Delta\left(\frac{P_n(n+1)-P_k(k+1)}{(n-k)\log(n-k+1)}\right)\right| = O(1);$$

$$\text{(ii)} \qquad \sum_{k=[n/2)}^{n-2} \left|\Delta\left(\frac{P_k}{k+1}\cdot\frac{1}{\log(n-k+1)}\right)\right| = O\left(\frac{P_n}{n}\right).$$

For proving (i) we observe that

$$\begin{aligned}
&\sum_{k=0}^{[n/2]-2} \left|\Delta\left(\frac{P_n(n+1)-P_k(k+1)}{(n-k)\log(n-k+1)}\right)\right| \\
&\quad\leqq \sum_{k=0}^{[n/2]-2} \frac{P_n(n+1)-P_k(k+1)}{(n-k)(n-k-1)\log(n-k+1)} \\
&\qquad + \sum_{k=0}^{[n/2]-2} \frac{P_{k+1}(k+2)-P_k(k+1)}{(n-k)\log(n-k+1)} \\
&\qquad + O\left(\sum_{k=0}^{[n/2]-2} \frac{P_n(n+1)-P_k(k+1)}{(n-k)^2\log^2(n-k+1)}\right) \\
&\quad = O\left(\sum_{k=0}^{[n/2]-2} \frac{1}{n-k}\right) + O\left[\sum_{k=0}^{[n/2]-2} \frac{P_{k+1}}{(n-k)\log(n-k+1)}\right] \\
&\qquad + O\left[\sum_{k=0}^{[n/2]-2} \frac{1}{(n-k)\log(n-k+1)}\right] \\
&\quad = O\left(\sum_{k=0}^{[n/2]-2} \frac{1}{(n-k)}\right) = O(1).
\end{aligned}$$

Again

$$\sum_{k=[n/2]}^{n-2} \left| \Delta\left(\frac{P_k}{k+1} \cdot \frac{1}{\log(n-k+1)}\right)\right|$$

$$= O\left(\sum_{k=[n/2]}^{n-2} \frac{P_k}{k+1} \cdot \frac{1}{(n-k)\log^2(n-k+1)}\right)$$

$$+ O\left(\sum_{k=[n/2]}^{n-2} \frac{P_k}{(k+1)^2 \log(n-k+1)}\right)$$

$$= O(P_n/n)\left(\sum_{k=[n/2]}^{n-2} \frac{1}{(n-k)\log^2(n-k+1)}\right) + O\left(\frac{P_n}{n^2}\right)\cdot(n)$$

$$= O(P_n/n).$$

This proves the lemma completely.

3. **Proof of the theorem.** Since

$$t_n = \frac{P_n u_0 + P_{n-1}u_1 + \cdots + P_0 u_n}{P_n}, \qquad \left(u_n = \frac{A_n(t)}{\log(n+1)}\right),$$

we have

$$t_n - t_{n-1} = \sum_{\nu=0}^{n-1}\left(\frac{P_\nu}{P_n} - \frac{P_{\nu-1}}{P_{n-1}}\right)u_{n-\nu}$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1}\right)u_{n-\nu}.$$

For the Fourier series of $f(t)$ at $t=k$,

$$A_n = \frac{2}{\pi}\int_0^\pi \phi(t)\cos nt dt$$

so that

$$t_n - t_{n-1} = \frac{2}{\pi}\int_0^\pi \phi(t)\left(\frac{1}{P_n P_{n-1}}\sum_{k=0}^{n-1}\left(\frac{P_n}{k+1} - \frac{P_k}{n+1}\right)\frac{\cos(n-k)t}{\log(n-k+1)}\right)dt.$$

Thus in order to prove the theorem, we have to establish that

$$\sum_n \left|\int_0^\pi \phi(t) g(n,t)dt\right| < \infty,$$

where

$$g(n,t) = \frac{1}{P_n P_{n-1}}\sum_{k=0}^{n-1}\left(\frac{P_n}{k+1} - \frac{P_k}{n+1}\right)\frac{\cos(n-k)t}{\log(n-k+1)}.$$

We observe that

$$\int_0^\pi \phi(t)g(n,t)dt = -\int_0^\pi \left(\int_0^t g(n,u)du\right) d\phi(t),$$

and

$$\sum_n \left|\int_0^\pi \left(\int_0^t g(n,u)du\right) d\phi(t)\right| \leqq \int_0^\pi |\, d\phi(t)\,| \left\{\sum_n \left|\int_0^t g(n,u)du\right|\right\}.$$

Since, by hypothesis, $\int_0^\pi |\, d\phi(t)\,| < \infty$, it suffices for our purpose to show that, uniformly for $0 < t < \pi$,

$$\sum_n \left|\int_0^t g(n,u)du\right| < \infty,$$

or, what is the same thing,

$$\sum \equiv \sum_n \left|\frac{1}{P_nP_{n-1}} \sum_{k=0}^{n-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1}\right) \frac{\sin(n-k)t}{(n-k)\log(n-k+1)}\right| < \infty.$$

Denoting $\tau = [1/t]$, we have

$$\begin{aligned}\sum \leqq \sum_1^\tau \frac{1}{P_nP_{n-1}} &\left|\sum_{k=0}^{n-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1}\right) \frac{\sin(n-k)t}{(n-k)\log(n-k+1)}\right| \\ &+ \sum_{\tau+1}^\infty \frac{1}{P_nP_{n-1}} \left|\sum_{k=0}^{[n/2]-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1}\right) \frac{\sin(n-k)t}{(n-k)\log(n-k+1)}\right| \\ &+ \sum_{\tau+1}^\infty \frac{1}{P_nP_{n-1}} \left|\sum_{[n/2]}^{n-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1}\right) \frac{\sin(n-k)t}{(n-k)\log(n-k+1)}\right|.\end{aligned}$$

Now since $|\sin(n-k)t| \leqq (n-k)t$ and $P_n(n+1) \geqq P_k(k+1)$ for $k \leqq n$, we have

$$\begin{aligned}\sum_1 &= \sum_1^\tau \frac{1}{P_nP_{n-1}} \left|\sum_{k=0}^{n-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1}\right) \frac{\sin(n-k)t}{(n-k)\log(n-k+1)}\right| \\ &\leqq At \sum_1^\tau \frac{1}{P_nP_{n-1}} \sum_{k=0}^{n-1} P_n/k+1 \\ &= At \sum_1^\tau P_nP_{n-1}/P_nP_{n-1} \\ &= O(1).\end{aligned} \tag{3.1}$$

By Abel's transformation and taking r to be a fixed number $> \pi$,

$$
\begin{aligned}
\sum_{2} &= \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n-k)t}{(n-k)\log(n-k+1)} \right| \\
&= \sum_{\tau+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]-1} \frac{P_n(n+1) - P_k(k+1)}{(n-k)\log(n-k+1)} \cdot \frac{\sin(n-k)t}{k+1} \right| \\
&= O\left[\sum_{\tau+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \left(\log \frac{r}{t} \right) \frac{P_n(n+1) - P_{[n/2]-1}\left(\left[\frac{n}{2} \right] \right)}{\left(\left[\frac{n}{2} \right] + 1 \right) \log\left(\left[\frac{n}{2} \right] + 2 \right)} \right] \\
&\quad + O\left[\sum_{\tau+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \left(\log \frac{r}{t} \right) \sum_{k=0}^{[n/2]-2} \left| \Delta \frac{P_n(n+1) - P_k(k+1)}{(n-k)\log(n-k+1)} \right| \right. \\
&= O\left(\log \frac{r}{t} \right) \sum_{\tau+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} = O\left(\frac{\log r/t}{P_\tau} \right) \\
&= O(1),
\end{aligned}
\tag{3.2}
$$

by using Lemmas 3 and 4 (i).

Since $1/(n+1)(n-k) = 1/(k+1)(n-k) - 1/(n+1)(k+1)$, we have

$$
\begin{aligned}
\sum_{3} &= \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n-k)t}{(n-k)\log(n-k+1)} \right| \\
&\leqq \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_n - P_k}{(k+1)} \cdot \frac{\sin(n-k)t}{(n-k)\log(n-k+1)} \right| \\
&\quad + \sum_{\tau+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_k}{k+1} \cdot \frac{\sin(n-k)t}{\log(n-k+1)} \right| \\
&= \sum_{31} + \sum_{32}, \text{ say.}
\end{aligned}
$$

Now since for $k \geqq [n/2]$, $P_n - P_k = O(1)$, we obtain

$$
\begin{aligned}
\sum_{31} &= \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{(P_n - P_k)}{k+1} \cdot \frac{\sin(n-k)t}{(n-k)\log(n-k+1)} \right| \\
&= O\left[\sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_n - P_k}{(k+1)} \cdot \frac{1}{(n-k)\log(n-k+1)} \right| \right] \\
&= O\left(\sum_{\tau+1}^{\infty} \frac{1}{n P_n P_{n-1}} \sum_{k=[n/2]}^{n-1} \frac{1}{(n-k)\log(n-k+1)} \right) \\
&= O\left(\sum_{\tau+1}^{\infty} \frac{\log\log n}{n \log^2 n} \right) = O(1).
\end{aligned}
\tag{3.3}
$$

Finally since $\sum_a^b \sin nt = O(1/t)$ for all a and b, we have by Abel's transformation

$$
\begin{aligned}
\sum_{32} &= \sum_{\tau+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_k}{k+1} \cdot \frac{\sin(n-k)t}{\log(n-k+1)} \right| \\
&= O\left(\sum_{\tau+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \sum_{k=[n/2]}^{n-2} \frac{1}{t} \left| \Delta\left(\frac{P_k}{k+1} \cdot \frac{1}{\log(n-k+1)} \right) \right| \right) \\
&\quad + O\left(\frac{1}{t}\right)\left(\sum_{\tau+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \cdot \frac{P_{n-1}}{n} \right) \\
&= O\left(\frac{1}{t}\right)\left(\sum_{\tau+1}^{\infty} \frac{1}{n^2 \log n} \right) \\
&= O\left(\frac{1}{t}\right)\left(\frac{1}{\tau \log \tau} \right) \\
&= O(1),
\end{aligned}
\tag{3.4}
$$

using Lemma 4 (ii).

Collecting (3.1), (3.2), (3.3), and (3.4), we see that the theorem is completely proved.

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References

1. G. H. Hardy and W. W. Rogosinski, *Notes on Fourier series*, IV. *Summability* (R_2), Proc. Cambridge Philos. Soc. vol. 43 (1947) pp. 10–25.

2. Leonard McFadden, *Absolute Nörlund summability*, Duke Math. J. vol. 9 (1942) pp. 168–207.

3. R. Mohanty, *On the absolute Riesz summability of a Fourier series and allied series*, Proc. London Math. Soc. ser. 2 vol. 52 (1951) pp. 295–320.

4. M. Riesz, *Sur l'equivalence de certaines méthodes de sommation*, Proc. London Math. Soc. vol. 22 (1924) pp. 412–419.

5. E. C. Titchmarsh, *The theory of functions*, Oxford, 1939, p. 440.

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