CERTAIN COLLECTIONS OF ARCS IN E^3

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1. **Introduction.** In considering upper semicontinuous decompositions of E^3, it is sometimes useful to know whether a given collection of continua can be transformed, by a homeomorphism of E^3 onto itself, into another collection which is simpler in some respects; for example, a collection of straight line intervals might be transformed into a collection of vertical intervals, or a collection of arcs into a collection of straight line intervals. It might also be useful to know conditions under which such a transformation can be effected by means of a particular type of homeomorphism of E^3 onto itself.

In this paper, the following questions of this type will be considered. Suppose α and β are horizontal planes and G is a continuous collection of mutually exclusive arcs, each of which is irreducible from α to β and no one of which contains two points of any horizontal plane, such that the sum of the elements of G is compact and intersects α in a totally disconnected set. Under what conditions is there a homeomorphism of E^3 onto itself which takes each element of G onto a vertical interval and does not change the z-coordinate of any point?

It is shown, with the aid of certain results due to Bing [**1**] and Fort [**5**], that such a transformation is not always possible, even when the elements of G are straight line intervals. The following condition is found to be necessary and sufficient for the existence of such a transformation (see §3 for definitions of unfamiliar terms): For every positive number ϵ there exists a finite set $K_1, K_2, \cdots, K_n$ of topological cylinders with bases on α and β such that (1) the solid cylinders determined by $K_1, K_2, \cdots, K_n$ are mutually exclusive, (2) each arc of G is enclosed by some K_i and (3) each K_i has horizontal diameter less than ϵ.

2. **Examples.** The decomposition given by Bing in [**1**] can be modified so that the collection of nondegenerate elements is of the type considered above. If there were a homeomorphism of E^3 onto itself carrying these arcs onto vertical intervals, then by [**2**, Theorem 5], the decomposition space would be homeomorphic to E^3; since this is not the case, there is no such homeomorphism.

A stronger example is furnished by Fort's modification [**5**] of Bing's example. This modification can be carried out in such a way

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that there exist four horizontal planes α, β, γ, δ with α above β, β above γ, and γ above δ, such that each nondegenerate element of the decomposition is the sum of three intervals g_1, g_2, g_3 with end points on α and β, β and γ, and γ and δ, respectively. Let G denote the collection of all nondegenerate elements of this decomposition and let G_1, G_2 and G_3 denote, respectively, the collection of all intervals lying in an element of G and having end points on α and β, the collection of all such intervals having end points on β and γ, and the collection of all those with end points on γ and δ. Suppose there is a homeomorphism of E^3 onto itself which does not change the z-coordinate of any point and which takes each element of G_1 onto a vertical interval. Then there is a homeomorphism f_1 of E^3 onto itself which is fixed on β and on all points below β and which takes each element of G_1 onto a vertical interval. From the symmetry of the construction of G, it follows that there is also a homeomorphism f_3 of E^3 onto itself which is fixed on γ and all points above γ and which takes each element of G_3 onto a vertical interval. Then f_3f_1 is a homeomorphism of E^3 onto itself which is fixed on β, γ and all points between β and γ and which takes each element of G_1+G_3 onto a vertical interval. The proof of the main theorem below shows that there is a homeomorphism f_2 of E^3 onto itself which is fixed on γ and all points below γ, does not change the z-coordinate of any point, takes each element of G_2 onto a vertical interval, and is such that if p and q are two points above β and on the same vertical line, then $f_2(p)$ and $f_2(q)$ are on the same vertical line. The transformation $f_2f_3f_1$ is a homeomorphism of E^3 onto itself which takes each element of G onto a vertical interval. But this is impossible since it implies, as before, that the decomposition space is homeomorphic to E^3. Hence the elements of G_1 cannot be transformed into a collection of vertical intervals by a homeomorphism of E^3 onto itself which does not change the z-coordinate of any point.

It is perhaps worth noting that if G_1' is the decomposition of E^3 whose only nondegenerate elements are the elements of G_1, then the decomposition space of G_1' is homeomorphic to E^3. This is a consequence of the following theorem, which is essentially proved in [2]. *If G is a monotone upper semicontinuous decomposition of E^3 such that* (1) *the set of nondegenerate elements of G is* 0-*dimensional in the decomposition space and* (2) *for every positive number ϵ and every open set U containing the sum of the nondegenerate elements of G, there is a homeomorphism of E^3 onto itself which is fixed on E^3-U and which takes each element of G into a set of diameter less than ϵ, then the decomposition space is homeomorphic to E^3.*

3. **Definitions.** If G is a collection of sets, then G^* will denote the sum of the elements of G; the collection G is said to *fill up* a point set M if $G^*=M$.

A subset K of E^3 will be called a *topological cylinder* provided there exist two parallel planes α_0 and α_1 and a continuous collection G of mutually exclusive arcs filling up K such that (1) each element of G is irreducible from α_0 to α_1, (2) no element of G contains two points of any plane parallel to α_0, and (3) $\alpha_0 \cdot K$ and $\alpha_1 \cdot K$ are simple closed curves. The planar disks bounded by $\alpha_0 \cdot K$ and $\alpha_1 \cdot K$ will be called the *bases* of K and the collection G will be called a *set of generators* for K. A topological cylinder plus its bases will be called a *closed topological cylinder* and a closed topological cylinder plus its interior will be called a *solid topological* cylinder.

If K is a topological cylinder with bases on the planes α_0 and α_1, then K is said to *enclose* a point set M provided that (1) each point of M lies either between the planes α_0 and α_1 or else on one of those planes and (2) if α is a plane parallel to α_0 or α_1 and intersecting M, then the simple closed curve $\alpha \cdot K$ encloses $\alpha \cdot M$ (i.e., $\alpha \cdot M$ is a subset of the bounded component of $\alpha - \alpha \cdot K$).

If K is a topological cylinder with horizontal bases, then $\max(\text{dia}(\alpha \cdot K))$, α a horizontal plane, will be called the *horizontal diameter* of K.

4. LEMMA 1. *Suppose A is the annulus bounded by the unit circle C_1 and the circle C_2 with center O and radius 2, and G is a collection of mutually exclusive arcs filling up A such that each arc of G has one end point on C_1 and the other on C_2 and no arc of G contains two points of any circle with center O. Then there exists an isotopy $\{F_t\}$, $0 \leqq t \leqq 1$, such that* (1) *for each t, F_t is a homeomorphism of A onto itself which does not change the distance from O of any point,* (2) *F_0 is the identity on A and* (3) *F_1 is a homeomorphism which takes each element of G into an interval lying on a line through O.*

PROOF. Let g_0 be an element of G. There is a continuous function $\phi(r)$, $1 \leqq r \leqq 2$, such that g_0 has the polar coordinate equation $\theta = \phi(r)$. For each t in $[0, 1/2]$ and each point (r, θ) of A, let $F_t^1(r, \theta) = (r, \theta - 2t \cdot \phi(r))$. Then $\{F_t^1\}$, $0 \leqq t \leqq 1/2$, is an isotopy on A and F_0^1 is the identity. If $(r, \theta) \in g_0$, then $F_{1/2}^1(r, \theta) = (r, 0)$, so $F_{1/2}^1$ takes g_0 onto the interval with end points $(1, 0)$ and $(2, 0)$.

Let $g_0' = F_{1/2}^1(g_0)$ and let G' denote the collection of all images under $F_{1/2}^1$ of elements of G. For each point p of A, let $\theta(p)$ be the smallest non-negative polar angle for p and let $\pi(p)$ denote the point of intersection of C_1 and the arc of G' containing p. For each t in $[1/2, 1]$

and each point $p=(r, \theta)$ of A, let $F_t^2(p)=(r, 2(1-t)\cdot\theta(p)+(2t-1)\cdot\theta(\pi(p)))$.

Since $\pi(p)$ is continuous on A and $\theta(p)$ is continuous on $A-g_0'$, F_t^2 is continuous on $A-g_0'$. By a direct argument, it can be shown that F_t^2 is also continuous at each point of g_0', so it is continuous on all of A. From the fact that if $\theta(p_1)<\theta(p_2)$, then $\theta(\pi(p_1))<\theta(\pi(p_2))$, it follows that F_t^2 is 1-1 and hence is a homeomorphism. It is easily verified that $\{F_t^2\}$, $1/2\leqq t\leqq 1$, is an isotopy. Clearly $F_{1/2}^2$ is the identity on A, and since for each p in A, $\theta(\pi(p))$ is a polar angle for $F_1^2(p)$ and $\pi(p)$ is constant on any element of G', F_1^2 takes each element of G' into an interval lying on a line through O.

For each t in $[0, 1/2]$, let $F_t=F_t^1$ and for each t in $[1/2, 1]$, let $F_t=F_t^2F_{1/2}^1$. Then $\{F_t\}$, $0\leqq t\leqq 1$, is an isotopy satisfying the desired conditions.

LEMMA 2. *Suppose K_1 and K_2 are right circular cylinders with horizontal bases such that K_2 encloses K_1. For $i=1, 2$, let G_i be a set of generators for K_i and let U_i denote the interior of the closed cylinder determined by K_i. Then there exists a continuous collection G of mutually exclusive arcs filling up the closure of U_2-U_1 such that no element of G contains two points of any horizontal plane and such that $G_1+G_2\subset G$.*

PROOF. Suppose K_i, $i=1, 2$, is represented in cylindrical coordinates by the equations $r=i$, $0\leqq z\leqq 1$. It follows from Lemma 1 that there is an isotopy $\{F_t^1\}$, $1\leqq t\leqq 3/2$, such that (1) for each t in $[1, 3/2]$, F_t^1 is a homeomorphism of K_1 onto itself which does not change the z-coordinate of any point, (2) $F_{3/2}^1$ is the identity on K_1 and (3) F_1^1 takes each element of G_1 onto a vertical interval. Similarly, there exists an isotopy $\{F_t^2\}$, $3/2\leqq t\leqq 2$, such that for each t in $[3/2, 2]$, F_t^2 is a homeomorphism of K_2 onto itself which does not change the z-coordinate of any point, (2) $F_{3/2}^2$ is the identity on K_2 and (3) F_2^2 takes each element of G_2' onto a vertical interval.

Let $M=\mathrm{Cl}(U_2-U_1)$ and for each point $p=(r, \theta, z)$ of M, let $F(p)$ be the point (r, θ', z), where θ' is such that if $r\leqq 3/2$, $F_r^1(1, \theta, z)=(1, \theta', z)$ and if $r\geqq 3/2$, $F_r^2(2, \theta, z)=(2, \theta', z)$. Then F is a homeomorphism of M onto itself which does not change the z-coordinate of any point. Since F agrees with F_1^1 on K_1 and with F_2^2 on K_2, it takes each element of G_1+G_2 onto a vertical interval.

Let G' denote the collection of all vertical intervals lying in M and having one end point on α_0 and the other on α_1 and let G denote the collection of all images under F^{-1} of elements of G'. Then G is a collection of mutually exclusive arcs fillng up M and satisfying the desired conditions.

LEMMA 3. *Suppose* $K_0, K_1, K_2, \cdots, K_n$ *are topological cylinders with bases on the horizontal planes* α_0 *and* α_1 *such that* K_0 *encloses* K_j $(j=1, 2, \cdots, n)$ *and such that no two of the solid cylinders determined by* $K_1, K_2, \cdots, K_n$ *have a point in common. If for* $j=0, 1, 2, \cdots, n$, G_j *is a set of generators for* K_j *and* U_j *is the interior of the closed cylinder determined by* K_j, *then there is a continuous collection* G *of mutually exclusive arcs filling up the closure of*

$$U_0 - (U_1 + U_2 + \cdots U_n)$$

such that (1) *no element of* G *contains two points of any horizontal plane and* (2) *each of* $G_0, G_1, \cdots, G_n$ *is a subcollection of* G.

PROOF. Let $K_0', \cdots, K_n'$ denote right circular cylinders with bases on α_0 and α_1 which are related in the same way as the correspondingly lettered topological cylinders $K_0, \cdots, K_n$. Let U_j', $j=0, 1, \cdots, n$, denote the interior of K_j' and let M and M' denote, respectively, the closures of $U_0 - (U_1 + \cdots + U_n)$ and $U_0' - (U_1' + \cdots + U_n')$. It follows from Theorem 1 and Lemma 2 of [4] and the remark following the proof of Theorem 5 of [3] that there is a homeomorphism h of M onto M' which does not change the z-coordinate of any point. For $j=0, 1, \cdots, n$, let G_j' denote the collection of all images under h of elements of G_j.

Let C_0 be a right circular cylinder with bases on α_0 and α_1 which is enclosed by K_0' and encloses each of $K_1', \cdots, K_n'$. Let $C_1, \cdots, C_n$ be right circular cylinders which determine mutually exclusive solid cylinders, such that C_j encloses K_j' and is enclosed by C_0. Let V_j denote the interior of C_j, let $M_0 = \mathrm{Cl}(U_0' - V_0)$ and for $j=1, 2, \cdots, n$, let $M_j = \mathrm{Cl}(V_j - U_j')$. It follows from Lemma 2 that, for $j=0, 1, \cdots, n$, there exists a continuous collection H_j of mutually exclusive arcs filling up M_j such that no element of H_j contains two points of any horizontal plane, $G_j' \subset H_i$, and every element of H_j which intersects C_j is a vertical interval. Let $H = H_1 + H_2 + \cdots + H_n$ and let G' denote the collection obtained by adding to H all vertical intervals with end points on α_0 and α_1 which intersect M'. Then the collection G of all images under h^{-1} of elements of G' satisfies the desired conditions.

THEOREM. *Suppose* α_0 *and* α_1 *are horizontal planes and* G *is a continuous collection of mutually exclusive arcs such that* (1) *each element of* G *is irreducible from* α_0 *to* α_1 *and no element of* G *contains two points of any horizontal plane, and* (2) G^* *is compact and intersects* α_0 *in a totally disconnected set. In order that there should exist a homeomorphism of* E^3 *onto itself which takes each element of* G *onto a vertical interval and* *does not change the z-coordinate of any point, it is necessary and sufficient that for every positive number* ϵ, *there exist a finite set* $K_1, K_2, \cdots, K_n$ *of topological cylinders with bases on* α_0 *and* α_1 *such that* (1) *the solid cylinders determined by* $K_1, K_2, \cdots, K_n$ *are mutually exclusive,* (2) *each arc of* G *is enclosed by some* K_i *and* (3) *each* K_i *has horizontal diameter less than* ϵ.

PROOF. 1. Suppose there is a homeomorphism h of E^3 onto itself which takes each element of G onto a vertical interval and does not change the z-coordinate of any point. Let G' denote the set of images under h of the elements of G and let K' be a vertical cylinder with bases on α_0 and α_1 which encloses G'^*.

Suppose ϵ is a positive number. Let S be a compact set containing the solid cylinder determined by K' in its interior. There is a positive number δ such that if p and q are points of S and $\rho(p, q) < \delta$, then $\rho(h^{-1}(p), h^{-1}(q)) < \epsilon$. Since $\alpha_0 \cdot G'^*$ is compact and totally disconnected, there exists a finite set $D_1, D_2, \cdots, D_n$ of mutually exclusive disks in α_0, each of diameter less than ϵ, such that every point of $\alpha_0 \cdot G'^*$ is in the interior of some D_i. Let K_i', $i = 1, 2, \cdots, n$, denote the topological cylinder having D_i as one of its bases and having its other base on α_1, which has a collection of vertical intervals as a set of generators. If $K_i = h^{-1}(K_i')$, then $K_1, K_2, \cdots, K_n$ satisfy the conditions of the theorem.

2. Suppose the condition is satisfied. Let K be a topological cylinder having a set of vertical generators, such that the bases of K are on α_0 and α_1 and K encloses G^*. By hypothesis, there exists a sequence $H_1, H_2, H_3, \cdots$ such that (1) for each n, H_n is a finite collection of cylinders each having one base on α_0 and the other on α_1, such that no two of the solid cylinders determined by the elements of H_n have a point in common, (2) K encloses each element of H_1 and for each n, each element of H_{n+1} is enclosed by some element of H_n, (3) for each n, each arc of G is enclosed by some element of H_n, and (4) for each n, each element of H_n has horizontal diameter less than $1/n$.

Let U denote the interior of the closed cylinder determined by K and for each n, let U_n denote the sum of the interiors of the closed cylinders determined by the elements of H_n.

Let G_0 be the set of vertical generators for K. It follows from Lemma 3 that there exists a continuous collection G_1 of mutually exclusive arcs filling up the closure of $U - U_1$ such that (1) each element of G_1 is irreducible from α_0 to α_1 and no element of G_1 contains two points of any horizontal plane and (2) $G_0 \subset G_1$ and each arc of G_1 which intersects an element of H_1 is a subset of that element. By applying Lemma 3 to each element of H_1, it can be shown that there is a continuous collection G_2 of mutually exclusive arcs filling up the closure of $U-U_2$, satisfying the first condition imposed on G_1 above and such that $G_1 \subset G_2$ and each arc of G_2 which intersects an element of H_2 is a subset of that element. By continuing this process, there may be obtained a sequence $G_1, G_2, G_3, \cdots$ such that (1) for each n, G_n is a continuous collection of mutually exclusive arcs filling up the closure of $U-U_n$ such that each element of G_n is irreducible from α_0 to α_1 and no element of G_n contains two points of any horizontal plane, and (2) for each n, $G_n \subset G_{n+1}$. Let $G' = G + G_1 + G_2 + \cdots$. Then G' is a continuous collection of mutually exclusive arcs filling up the solid cylinder determined by K, each element of G' is irreducible from α_0 to α_1, no element of G' contains two points of any horizontal plane, and each element of G' which intersects K is a vertical interval.

Let M denote the solid cylinder determined by K. For each point p of M, let $f(p)$ be that point q on the horizontal plane containing p such that the projection of q onto α_0 is an end point of the arc of G' containing p. Then f is a homeomorphism of M onto itself which is fixed on K and on $M \cdot \alpha_0$, does not change the z-coordinate of any point, and takes each element of G' onto a vertical interval. Let F be the function which agrees with f on M, leaves fixed each point of $E^3 - M$ not lying directly above a point of M, and is such that if p is a point of $E^3 - M$ lying directly above the point q of $\alpha_1 \cdot M$ (supposing α_1 is above α_0), then $F(p)$ is the point with the same z-coordinate as p which lies directly above the point $f(q)$. Then F is a homeomorphism of E^3 onto itself which satisfies all the desired conditions.

References

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