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AN EXTREMAL PROBLEM FOR POLYNOMIALS

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PROBLEM. Consider the class of nth order polynomials $\{f(z)\}$ such that $f(1)=0$, $|f(z)|\leqq 1$ for $|z|=1$. From this class select that polynomial for which

$$\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^2 d\theta \text{ is greatest.}$$

For the solution we require the following

LEMMA. *Let* $h(z)=\sum_{-N}^{N} h_n z^n$, $(\bar{h}_n = h_{-n})$. *Then there exists a polynomial* $f(z)$ *of degree* N *such that* $h(z)=|f(z)|^2$ *for* $|z|=1$ *if and only if* $h(z)\geqq 0$ *for* $|z|=1$. *Proof is available in reference* [1].

The function $1-|f(z)|^2$ (with $\bar{z}$ replaced by $1/z$) satisfies the conditions of the lemma for any $f(z)$ that satisfies the conditions of the problem. Thus, we can write, $1-|f(z)|^2=|g(z)|^2$, where $g(z)$ satisfies the conditions that $|g(z)|\leqq 1$ for $|z|=1$ and $|g(1)|=1$. (Without real loss of generality, we take this last to mean $g(1)=1$.)

In addition, for $f(z)$ to solve the problem, the associated $g(z)$ must minimize the integral

$$\frac{1}{2\pi}\int_0^{2\pi}|g(e^{i\theta})|^2 d\theta.$$

Writing $g(z)=\sum_0^N g_n z^n$, we see that we are seeking to minimize the quantity $\sum_0^N |g_n|^2$ subject to the constraint that $\sum_0^N g_n=1$. A straightforward application of the Schwarz Inequality yields: $1=\sum_0^N g_n \leqq (\sum_0^N |g_n|^2)^{1/2}(N+1)^{1/2}$. The sum-of-squares is smallest when we set $g_n=1/(N+1)$, and obtain for the corresponding $g(z)$,

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$$g(z) = \frac{1}{N+1} \sum_{0}^{N} z^n.$$

This well known function is equal to unity for $z=1$ and vanishes when z is any of the other $N+1$st roots of unity. It is easy to show that

$$\left| g(e^{i\theta}) \right| = \left| \frac{\sin \frac{N+1}{2} \theta}{(N+1) \sin \theta/2} \right|, \qquad -\pi \leqq \theta \leqq \pi.$$

From this it is seen that $|g(z)| \leqq 1$ for $|z| = 1$, so that this constraint is satisfied even though we did not impose it in determining $g(z)$. As a consequence, the function $1 - |g(z)|^2$ (with $1/z$ set for $\bar{z}$) satisfies the conditions of the lemma so that the associated function, $f(z)$, is the solution to the problem.

One obtains a fair idea of the nature of $f(z)$ from the observations that it never passes outside the unit circle, it passes through the origin for $z=1$, and it is tangent on the inside to the unit circle at all the other $N+1$st roots of unity. The value of the integral being maximized is $N/(N+1)$.

The method of computation of the Nth order f is given essentially in the proof of the lemma:

Solve the reciprocal equation $1-g(z)g(1/z)=0$, *and, from each pair of reciprocal roots select one member. Then f is that Nth order polynomial having these selected quantities for roots.*

f must also be properly normalized, of course. To remove an irrelevant ambiguity, we may specify that none of the roots should lie outside the unit circle.

For example:

$$f_0 = 0,$$
$$f_1 = (z-1)/2,$$
$$f_2 = ((1+3^{1/2})z^2 - 2z + (1-3^{1/2}))/(18)^{1/2}.$$

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Reference

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