BASIC SEQUENCES IN THE SPACE OF MEASURABLE FUNCTIONS

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1. In a topological vector space X, a basic sequence $\{x_n\}$ is one whose finite linear combinations are dense in X. In a recent work, [1], A. A. Talalyan has observed that the space of measurable functions has a distinctly different character, with respect to the behavior of basic sequences, from, for example, the L_p spaces, $p \geqq 1$.

A striking result of Talalyan is the fact that if $\{\phi_n\}$ is basic, i.e., for every measurable ϕ, there are finite linear combinations of the ϕ_n which converge almost everywhere to ϕ, then if any finite number of functions is deleted from $\{\phi_n\}$, the remaining sequence is basic. This readily implies the existence of universal expansions, and the existence of a subsequence $\{\phi_{n_k}\}$ which is basic even though the complement of the sequence $\{n_k\}$ is infinite.

The proof given by Talalyan necessitates the use of considerable machinery from the theory of orthonormal systems in L_2, and is quite involved. Our purpose is to show that the result follows almost immediately from the fact that the space M of measurable functions, with the topology of convergence in measure, has a trivial dual.

2. With the topology of convergence in measure, the space M of equivalence classes of measurable functions on $[0, 1]$ is a metrizable topological vector space. The following is well known, [2], but the proof is very short and is included for completeness.

LEMMA 1. *The dual M' of M consists only of* 0.

PROOF. Let f be a continuous linear functional on M. If $f(x) \neq 0$ for some $x = x(t) \in M$, then, since the step functions are dense in M, there is a sequence $\{I_n\}$ of intervals, whose lengths converge to zero, such that $f(\chi_{I_n}) \neq 0$. There are then constants k_n such that $f(k_n\chi_{I_n}) = 1$ for every n. But $\lim_n k_n\chi_{I_n} = 0$ in M.

We also need

LEMMA 2. *If the closure of the vector space generated by $\phi_1, \phi_2, \cdots$ is M, and the closure of the vector space generated by $\phi_2, \phi_3, \cdots$ is $X \subset M$, then M is the span of X and ϕ_1.*

PROOF. If $\phi_1 \in X$, the result is obvious. Suppose $\phi_1 \notin X$. Let $x \in M$. There are $y_n \in X$, a_n real, such that $\lim_n (a_n\phi_1 + y_n) = x$. If $\{a_n\}$ had a subsequence $\{a_{n_k}\}$ converging to infinity, then

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$\{\phi_1+y_{n_k}/a_{n_k}\}$ would converge to zero, so that $\phi_1=-\lim_k y_{n_k}/a_{n_k}$ would belong to X. This being false, $\{a_n\}$ is bounded and has a convergent subsequence $\{a_{m_k}\}$. Let $a=\lim_k a_{m_k}$. Now $\lim_k (a_{m_k}\phi_1+y_{m_k}) =x$ implies $\{y_{m_k}\}$ converges to $y\in X$, and thus $a\phi_1+y=x$.

COROLLARY 1. $X=M$.

PROOF. Suppose $X\neq M$. In view of Lemma 2, there exists a continuous linear functional on M which vanishes on X and is 1 at ϕ_1. This contradicts Lemma 1.

3. Let $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ be a basic sequence in M. By Corollary 1, the closure of the vector space generated by $\phi_2, \phi_3, \cdots, \phi_n, \cdots$ is also M. Indeed, for every k, the sequence $\phi_k, \phi_{k+1}, \cdots, \phi_n, \cdots$ is basic. We thus have the

THEOREM. *If $\{\phi_n\}$ is a basic sequence in M, it remains basic if any finite subset is deleted.*

4. This theorem has two interesting corollaries. A series $\sum_{n=1}^{\infty} a_n\phi_n$ is called *universal* in a space X if for every $x\in X$, a subsequence of the sequence of partial sums of the series converges to x.

COROLLARY 2. *If $\{\phi_n\}$ is a basic sequence in M, there is a sequence $\{a_n\}$ such that the series $\sum_{n=1}^{\infty} a_n\phi_n$ is universal in M.*

PROOF. The space M is metrizable with distance

$$d(x, y) = \int_0^1 \frac{|x(t) - y(t)|}{1 + |x(t) - y(t)|}\, dt.$$

Let $\{x_n\}$ be a countable dense subset of M. Choose a sequence of positive real numbers $\{\epsilon_n\}$ which converges to zero. There exists a linear combination $\sum_{n=1}^{k_1} a_n\phi_n$ such that $d(x_1, \sum_{n=1}^{k_1} a_n\phi_n)<\epsilon_1$. In view of our theorem, there is a combination $\sum_{k_1+1}^{k_2} a_n\phi_n$ which approximates $x_2-S_{k_1}$ to within ϵ_2. Similarly there is $\sum_{k_2+1}^{k_3} a_n\phi_n$ approximating $x_1-S_{k_2}$ to within ϵ_3. We then determine $\sum_{k_3+1}^{k_4} a_n\phi_n$ and $\sum_{k_4+1}^{k_5} a_n\phi_n$ such that $d(x_2 - S_{k_3}, \sum_{k_3+1}^{k_4} a_n\phi_n) < \epsilon_4$ and $d(x_3-S_{k_4}, \sum_{k_4+1}^{k_5} a_n\phi_n)<\epsilon_5$. We now begin again with x_1 and continue this process indefinitely, approximating ever-lengthening chains. It is clear from the construction that for every $x\in M$ there is a subsequence of the sequence of partial sums of $\sum_{n=1}^{\infty} a_n\phi_n$ which converges to x.

Another consequence of the theorem is

COROLLARY 3. *If $\{\phi_n\}$ is a basic sequence in M, there is a subsequence obtained by an infinite number of deletions which is also basic.*

PROOF. Consider a sequence $\{\epsilon_n\}$ as described above. Delete ϕ_1 from $\{\phi_n\}$. ϕ_1 can be approximated to within ϵ_1 by a combination of ϕ_n with highest index n_2; delete ϕ_{n_2+1}. We note that ϕ_1 and ϕ_{n_2+1} can be approximated by the remaining functions to within ϵ_2. Let n_3 be the largest index used in these approximations. Clearly we may take $n_3>n_2$. Delete ϕ_{n_3+1}. Now approximate the three functions deleted to within ϵ_3. As before, we take the largest index used, n_4, to be greater than n_3. Delete ϕ_{n_4+1} and repeat this process. It is clear that the subsequence which remains is basic since the deleted elements are approximable by finite linear combinations of the remaining functions.

5. **Remark.** We note that the class M can be widened to include the extended real valued functions, i.e., those which take on infinite values on sets of positive measure.

This is clearly a consequence of an extension of almost everywhere convergence to include the case of sequences which diverge to $+\infty$ or $-\infty$ on sets of positive measure and the fact that infinite valued measurable functions are the limits, in this sense, of finite valued ones.

REFERENCES

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