A THEOREM CONCERNING SIX POINTS

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1. **Introduction and statement of results.** "Let three points be specified on a line. Then one of the points is in the interior of the line segment joining the other two, and one of the points is exterior to the line segment joining the other two." This elementary statement concerning the ordering of three points on a line is capable of various extensions. Thus, e.g., it is easy to prove the following: Let $n+2$ points which are not all on an n-sphere be specified in E_n, with some $n+1$ of them linearly independent. Then one of the points must be in the interior of the n-sphere passing through the other $n+1$ points, and one of the points must be exterior to the n-sphere passing through the other $n+1$ points.

In this paper we consider the following problem: Let six points be specified in a plane, no three collinear and not all on a conic. Must some one of these points be in the "interior" of the conic (i.e. in a convex component of the plane bounded by the conic section) passing through the remaining five, and must some one of the points be "exterior" to the conic (i.e. in the nonconvex component of the plane bounded by the conic) passing through the other five? This question cannot always be answered in the affirmative. We shall see that it is impossible for each point to be outside the conic through the other five, but it is possible for each point to be inside the conic through the other five.

Before stating our result precisely we make the following definition: Let five points no three collinear be specified having either five or three of these points on the boundary β of their convex hull. If β is a pentagon denote its vertices by A, B, C, D, E taken cyclically about β; if β is a triangle denote the two interior points by D and E and the other three by A, B, C such that the half-lines $DE^{\rightarrow}$ and $ED^{\rightarrow}$ intersect the sides AB and BC of β respectively (here and below "$XY^{\rightarrow}$" designates the (closed) half-line which is bounded by X and contains Y, X and Y being noncoincident points). Then the intersection of the open angular regions CAD, DBE, ACE each of opening less than 180° is called the *nucleus* of the 5-point configuration and denoted by $N(ABCDE)$.

We first prove the following lemma:

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LEMMA 1. *Let a triangle $T_1(ABC)$ (A, B, C are the vertices) contain a triangle $T_2(DEF)$ in its interior, no three elements of T: $\{A, B, C, D, E, F\}$ collinear. If one of the vertices of T_2 is inside the nucleus of the other elements of T, then each vertex of T_2 is inside the nucleus of the other elements of T.*

Throughout this paper the symbol S_6 will denote a set of six points in a plane (E_2), no three of the points collinear and not all on a conic; "b" will denote the boundary of the convex-hull of S_6.

If one element of S_6 is in the interior of the conic through the other five, and one element of S_6 is in the exterior of the conic through the other five then we say that S_6 is *simply-selfcovering*. If each element of S_6 is in the interior of the conic through the other five then we say that S_6 is *completely-selfcovering*.

We prove the following theorem:

THEOREM. (i) *If b has 4 or 6 vertices, then S_6 is simply-selfcovering.* (ii) *If b has 3 or 5 vertices, then S_6 is completely- or simply-selfcovering accordingly as any element of S_6 not on b belongs to, or does not belong to the nucleus of the other five.*

2. PROOF OF LEMMA 1. We let

(i) $l[PQ]$ denote the set of points on the line passing through the points P and Q;

(ii) $[PQ]$ denote all points of $l[PQ]$ between (but not including) P and Q;

(iii) $[PQR]$ denote the set of all points of the plane interior to (but not including) the triangle with vertices P, Q, R;

(iv) $[PQRS]$ denote the set of all points of the plane interior to (but not including) the simply closed quadrilateral with vertices P, Q, R, S taken cyclically about it;

(v) $\langle QPR\rangle$ (P, Q, R not collinear) denote the convex open set of all points of the plane bounded by $PQ^{\rightarrow}$ and $PR^{\rightarrow}$.

Since no three vertices of T are collinear, we may assume that the vertices of T_1 are labelled such that the half-lines $DE^{\rightarrow}$ and $ED^{\rightarrow}$ intersect $[AB]$ and $[BC]$ respectively. Also, let $F\in N(ABCDE)$.

(1) Since $F\in N(ABCDE)\subset[ECA]$, $EF^{\rightarrow}$ intersects $[CA]$. Let $l[AE]$ intersect $[BC]$ in J; let $l[BE]$ intersect $[AC]$ in K; since $F\in N(ABCDE)\subset[CKEJ]$, $FE^{\rightarrow}$ intersects $[AB]$.

(2) Since $ED^{\rightarrow}$ intersects $[BC]$, $D\in[BCE]\subset\langle BCE\rangle$.

(3) Since $F\in[CEK]$, and $K\in BE^{\rightarrow}$, $BF^{\rightarrow}$ separates D and E and intersects $[CE]$ in a point, say H. Thus, $D\in[CBH]\subset\langle CBF\rangle$.

(4) Let $CD^{\rightarrow}$ and $BD^{\rightarrow}$ intersect $[BA]$ and $[CA]$ in Z and Y respectively. Since $ED^{\rightarrow}$ intersects $[BC]$,

$$E \in [ZDYA] \subset [ZCA] = [CDA] + [DA] + [DAZ].$$

As $[CDEA]$ is strongly convex and can be partitioned thus: $[CDEA] = [CDA]+[DA]+[DEA]$, we have $E\in[DAZ]$. Since $F\in[CDA]$, $[AD]$ intersects $[EF]$, and therefore $D\in\langle EAF\rangle$.

Thus, in view of (1), (2), (3), (4) (above this section), $D \in N(ABCEF)$.

In similar manner we can prove that $E\in N(ABCDF)$.

3. **"Projectively-cyclic" ordering of five points.** Let S_5 denote a set of five points in a (Euclidean) plane π, no three of the points being collinear. A conic Γ is uniquely determined by the elements of S_5. Let π be "closed" by adjoining the "line at infinity," and let us denote the projective plane thus obtained by π_p. Let Γ' be the conic in π_p such that $\Gamma' \supset \Gamma$. An ordering $(Q_1, Q_2, Q_3, Q_4, Q_5)$ of S_5 will be referred to as *projectively-cyclic* if Γ' partitions into abutting but non-overlapping arcs

$$Q_1Q_2,\ Q_2Q_3,\ Q_3Q_4,\ Q_4Q_5,\ Q_5Q_1.$$

REMARK. Thus, e.g. the points

$$(-1, 0),\ (-2, 1)\ (1, 0),\ (2, 1),\ (-2, -1)$$

on $x^2-y^2=1$ are projectively-cyclic in the stated order. If we denote the branches of $x^2-y^2=1$ by B_1 and B_2, then the element of $S_5\cdot B_1$ with largest ordinate is "adjacent" to the element of $S_5\cdot B_2$ with smallest ordinate (cf. [1]).

We now show *how to order the elements of S_5 so that the ordering is projectively-cyclic on the conic through them.*

Let β be the boundary of the convex hull of S_5.

CASE I. *Suppose β is a pentagon.* In this case the points of S_5 are on an ellipse, parabola or one branch of a hyperbola. (For, if they fell on both branches of a hyperbola, β could not be a pentagon.) Then, any cyclic ordering $(Q_1, Q_2, Q_3, Q_4, Q_5)$ of the vertices of β (i.e. Q_iQ_{i+1} are the edges of β, $Q_6\equiv Q_1$) is a projectively-cyclic ordering of S_5.

CASE II. *Suppose β is a quadrilateral.* Then the elements of S_5 obviously cannot fall on an ellipse, parabola or one branch of a hyperbola. Furthermore, it is not possible for one element of S_5 to be on one branch of a hyperbola and the remaining four on the other, for, in this case, β would be a triangle. Thus two elements of S_5 must be on one branch of a hyperbola, and three on the other. Now, let O be the intersection of the diagonals of β; let C and D be the adjacent vertices of β such that $[OCD]$ contains the element E of S_5 which is not on β; let β be the polygon (A, B, C, D). There are ten ways of partitioning $S_5 = S_2^{(i)} + S_3^{(i)}$ $(i = 1, 2, \cdots, 10)$, $S_2^{(i)}$ and $S_3^{(i)}$ containing two and three elements respectively. If for each case we assume that $S_2^{(i)}$ is on one branch of a hyperbola and $S_3^{(i)}$ is on the other, then for nine values of i we obtain a contradiction of the fact that a straight line can cut a conic section in at most two points. The remaining case, where A and B are on one branch and C, E, D on the other, must therefore hold. Furthermore, from the same fact it may be deduced that E is "between" C and D on the branch on which they lie. Thus, the ordering (A, B, D, E, C) (or any cyclic permutation) is projectively-cyclic.

Case III. *Suppose β is a triangle.* Then S_5 cannot fall on an ellipse, parabola or one branch of a hyperbola; or, on two branches of a hyperbola with two and three points on separate branches. The only remaining case is where one point of S_5 is on one branch of a hyperbola and four on the other. Let X and Y be the points of S_5 not on β. Then one of the points of S_5, say P, must be on one side of the line $l(X, Y)$ and two, say Q and R, on the other. Let the half-line $YX^{\rightarrow}$ intersect β in the edge PQ. Then P is on one branch of a hyperbola and R, Y, X, Q on the other (this is again obtained by considering the five possibilities and eliminating four of them from the fact that a line cuts a conic in at most two points). Furthermore, since the polygon (R, Y, X, Q) is convex, R, Y, X, Q must be the ordering of these points on the branch on which they lie. Thus, the ordering (P, R, Y, X, Q) is projectively-cyclic.

Let $\Delta(Q, R, S, T, U, V)$ be the determinant with ith row

$$x_i^2 \quad y_i^2 \quad x_i y_i \quad x_i \quad y_i \quad 1$$

where (x_i, y_i) are the coordinates of the point in the ith position from the left in (the parentheses of) $\Delta(\ )$.

Lemma 2. *Let the ordered point set $(W_1, \cdots, W_5)$ (no three collinear) in E_2 be projectively-cyclic, and let Γ denote the conic through them. Then $\Delta(P, W_1, \cdots, W_5)$ is greater than, equal to or less than zero accordingly as P is inside, on or outside Γ, and conversely.*

Proof. Since translations, rotations and proper dilatations $(x' = \alpha x, y' = \gamma y, \alpha > 0, \gamma > 0)$ do not alter the sign of $\Delta(P, W_1, \cdots, W_5)$, we may assume that Γ is in standard position with (absolute values of) conic constants conveniently chosen. Furthermore, if W_i is moved along Γ to W_i' and at no point of this path does W_i cross its adjacent points W_{i-1} or W_{i+1} $(W_0 = W_5,\ W_6 = W_1)$ then the sign of $\Delta(P, W_1, \cdots, W_5)$ is unaltered. Thus, if the points $W_1, \cdots, W_5$

are moved along Γ to new positions but at no instant does W_i coincide with W_j $(i \neq j)$ then the sign of $\Delta(P, W_1, \cdots, W_5)$ remains unaltered. Verification of the sign of $\Delta(P, W_1, \cdots, W_5)$ for simple conics and conveniently chosen points $W_1, \cdots, W_5$ on them, for Cases I, II, III above, completes the proof of the lemma.

DEFINITION. Let $S_6 \equiv \{A, B, C, D, E, F\}$. We shall say that A and B are *similarly oriented* if there are ordered sets $(Z_1, \cdots, Z_4)$ and $(Z_{i_1}, \cdots, Z_{i_4})(\{Z_1, \cdots, Z_4\} \equiv \{C, D, E, F\})$ each obtainable from the other by an *even* number of adjacent interchanges such that $(A, Z_1, \cdots, Z_4)$ and $(B, Z_{i_1}, \cdots, Z_{i_4})$ are projectively-cyclic.

LEMMA 3. *If S_6 contains two elements which are similarly oriented, then it is simply-selfcovering.*

PROOF. Let A and B belong to S_6 and be similarly oriented. Then (A, V_1, V_2, V_3, V_4) and $(B, V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4})$ are projectively-cyclic, where $(V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4})$ can be obtained from (V_1, V_2, V_3, V_4) by an even number of adjacent interchanges $(\{A, B, V_1, V_2, V_3, V_4\} \equiv S_6)$. But

$$\begin{aligned}\Delta(A, B, V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}) &= \Delta(A, B, V_1, V_2, V_3, V_4) \\ &= -\Delta(B, A, V_1, V_2, V_3, V_4).\end{aligned}$$

By Lemma 2 if A is inside the conic Γ_1 through $B, V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}$ (it can't be on Γ_1 by definition of S_6), then $\Delta(B, A, V_1, V_2, V_3, V_4) < 0$, and B is therefore outside the conic Γ_2 through A, V_1, V_2, V_3, V_4. Similarly, if A is outside Γ_1, then B is inside Γ_2.

4. PROOF OF THE THEOREM. Let $S_6 = \{U, V, W, X, Y, Z\}$.

CASE I. Suppose b is a hexagon. Then any two adjacent vertices of b are similarly oriented. By Lemma 3, S_6 is simply-selfcovering.

CASE II. Suppose b is a pentagon $(UVWXY)$. (i) $Z \notin N(UVWXY)$. Then Z is outside a convex quadrilateral $(CDEF)$, $\{C, D, E, F\} \subset \{U, V, W, X, Y\}$. Let $B \in \{U, V, W, X, Y\}$, $B \notin \{C, D, E, F\}$. Then B and Z are similarly oriented.

(ii) $Z \in N(UVWXY)$. Z is obviously inside the conic through U, V, W, X, Y. We show that U is inside the conic through V, W, X, Y, Z: Let A be a point on the half-line $WV^{\rightarrow}$ with V between W and A. Let J be a point on $XY^{\rightarrow}$ with Y between X and J. If $VW^{\rightarrow}$ and $YX^{\rightarrow}$ intersect in a point (or if $l(VW)$ and $l(XY)$ are parallel) then U is inside the (open) convex region bounded by $VA^{\rightarrow} + [VY] + YJ^{\rightarrow}$ which is in the interior of the hyperbola Γ passing through V, Z, Y, W, X (cf. Case II, §3). If $VA^{\rightarrow}$ and $YJ^{\rightarrow}$ intersect in a point I, then U belongs to $[IVY]$ (since b is convex and no three elements of S_6 are collinear) which is in the interior of Γ. In similar manner it may be shown that V, W, X and Y are each in the interior of the conic through the other five elements of S_6.

CASE III. Suppose b is a quadrilateral $(WXYZ)$. Let the diagonals $[WY]$ and $[XZ]$ intersect in the point O.

(i) If U and V are both in the same one of the triangular regions

$$[OWX],\ [OXY],\ [OYZ],\ [OZW], \tag{4.1}$$

then they are similarly oriented (cf. Case II of §3).

(ii) Suppose U and V do not fall in the same element of (4.1), and the boundaries of the elements of (4.1) in which they do fall have only the point O in common. Suppose e.g. that $U\in[OZW]$ and $V\in[OXY]$. Then (U, W, Y, X, Z) and (V, Y, W, Z, X) are projectively-cyclic orderings (cf. Case II, §3). But by an even number of adjacent interchanges: $VYWZX\rightarrow VWYZX\rightarrow VWYXZ$. Thus U and V are similarly oriented. The other cases of (ii) are handled similarly.

(iii) Suppose U and V do not fall in the same element of (4.1) and that the boundaries of the elements of (4.1) in which they do fall have an edge in common. Thus, e.g., let $U\in[OWX]$ and $V\in[OXY]$. If the boundary of the convex hull of W, Z, Y, V, U has these five points as its vertices, then X and Z are similarly oriented since (Z, W, U, V, Y) and (X, W, U, V, Y) are projectively-cyclic (cf. Cases I and III of §3). If the boundary of the convex hull of W, Z, Y, V, U has only four vertices e.g. W, V, Y, Z, then Z and X are similarly oriented since (Z, V, U, W, Y) and (X, V, U, W, Y) are projectively-cyclic (cf. Cases II and III of §3). The other cases of (iii) are handled similarly.

CASE IV. Suppose b is a triangle (XYZ). Let U and V be designated so that U and V make with X and Z a convex quadrilateral $(UZXV)$. Let

$$\begin{array}{lll} [ZV]\cdot[XU]=K, & [YU]\cdot[ZV]=L, & [YU]\cdot[ZX]=M, \\ [YV]\cdot[ZX]=N, & [YV]\cdot[UX]=P, & [XV]\cdot[YK]=Q, \\ [ZU]\cdot[YK]=R, & [XV]\cdot[YZ]=S, & [ZU]\cdot[YX]=T. \end{array}$$

(i) Suppose $W\notin N(UVXYZ)$. (a) If $W\in[ZLM]$ then W and Z are similarly oriented. If $W\in[XPN]$ then W and X are similarly oriented. (b) If $W\in[ZKQS]$, then W and U are similarly oriented. If $W\in[XKRT]$, then W and V are similarly oriented. (c) If $W\in[SQY]$, then W and Y are similarly oriented. If $W\in[YRT]$, then W and Y are similarly oriented.

(ii) Suppose $W \in N(UVXYZ)$. We show that each element of S_6 is inside the conic through the other five. (a) W is inside the hyperbola Γ (cf. Case III, §3) passing through Z, U, V, X, Y (since Z, U, V, X are all on one branch of Γ, $[ZUVX]$ is convex and $Z \in [ZUVX]$). In similar manner using Lemma 1, it may be shown that U and V are each inside the conic through the other five elements of S_6. (b) We now show that Y is inside the hyperbola θ through X, Z, U, V, W. By Case II of §3, Z, W, X are on one branch θ_1 of θ and U, V on θ_2 the other branch. Now, $YU^{\rightarrow}$ intersects $[ZW]$ in a point G which is inside θ_1, where U is between Y and G. Therefore $YU^{\rightarrow}$ intersects θ_1 in a point H, where U is between Y and H. Thus Y is inside θ_2. In similar manner using Lemma 1, it may be shown that X and Z are each inside the conic through the other five elements of S_6.

Reference

1. O. Veblen and J. W. Young, *Projective geometry*, vol. II, Ginn, 1918, pp. 67–69.

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