ON LIE ALGEBRAS OF CLASSICAL TYPE

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G. B. Seligman has proved in [1] the following result: If $\mathfrak{L}$ is a simple restricted Lie algebra over an algebraically closed field of characteristic $p>7$, and if $\mathfrak{L}$ has a restricted representation with nondegenerate trace form, then $\mathfrak{L}$ is of classical type. By an algebra of classical type is meant an analogue over a field of characteristic p of one of the simple Lie algebras (including the five exceptional algebras) of characteristic 0; for the precise statement, see [1]. We shall show here that the above result of Seligman may be proved without the assumption of restrictedness of the algebra and its representation.

We begin with a lemma on matrices. Let $\mathfrak{M}_n(\mathfrak{F})$ denote the space of all $n\times n$ matrices over a field $\mathfrak{F}$, and consider this as a Lie algebra (under commutation) over $\mathfrak{F}$.

LEMMA. *Let $\mathfrak{L}$ be a (Lie) subalgebra of $\mathfrak{M}_n(\mathfrak{F})$, such that the trace form $f(A, B)=\mathrm{tr}(AB)$ is nondegenerate on $\mathfrak{L}$, and let $\mathfrak{N}$ be the normalizer in $\mathfrak{M}_n(\mathfrak{F})$ of $\mathfrak{L}$. Then $\mathfrak{L}$ is a direct summand of $\mathfrak{N}$.*

PROOF. Let $\mathfrak{Q}$ be the set of all D in $\mathfrak{N}$ such that $f(A, D)=0$ for all A in $\mathfrak{L}$. Every C in $\mathfrak{N}$ defines a linear functional $A\rightarrow f(A, C)$ on $\mathfrak{L}$, so by the nondegeneracy of f on $\mathfrak{L}$, there is a $B=B(C)$ in $\mathfrak{L}$ such that $f(A, B) = f(A, C)$ for all A in $\mathfrak{L}$. But then $C-B\in\mathfrak{Q}$ and $C=B+(C-B)$, so $\mathfrak{N}=\mathfrak{L}+\mathfrak{Q}$. Furthermore this sum is a vector space direct sum, since $\mathfrak{L}\cap\mathfrak{Q}=0$ by the nondegeneracy of f on $\mathfrak{L}$. But $\mathfrak{L}$ is an ideal of $\mathfrak{N}$ by the definition of normalizer, and $\mathfrak{Q}$ is ideal of $\mathfrak{N}$ since $f(A, [BC])=f([AB], C)$ for all A, B, C. Thus the lemma is proved.

By a *representation form* on a Lie algebra $\mathfrak{L}$ we shall mean a bilinear form f on $\mathfrak{L}$ for which there is a representation $S\colon x\rightarrow S_x=S(x)$ of $\mathfrak{L}$ such that $f(x, y)=\mathrm{tr}(S_xS_y)$ for all x, y in $\mathfrak{L}$.

THEOREM 1. *Let $\mathfrak{L}$ be a Lie algebra of characteristic p with a nondegenerate representation form f. Then $\mathfrak{L}$ is restricted.*

PROOF. Let S be a representation of $\mathfrak{L}$ giving rise to f. If $S_x=0$ then $f(x, \mathfrak{L})=0$, so $x=0$. Hence S is an isomorphism, so it suffices to prove that the image $S_\mathfrak{L}$ of $\mathfrak{L}$ is restricted. For any A and C in $S_\mathfrak{L}$,

$$[AC^p] = [\cdots[A\overbrace{C]\cdots C}^{p}]\in S_\mathfrak{L}.$$

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Hence by the lemma, there is a B in $S_{\mathfrak{L}}$ such that $[AC^p]=[AB]$ for all A in $S_{\mathfrak{L}}$. Since a Lie algebra is restricted if the pth power of every inner derivation is inner, it folllws that $S_{\mathfrak{L}}$ and $\mathfrak{L}$ are restricted.

We shall write R for the (right) adjoint representation.

THEOREM 2. *Let $\mathfrak{L}$ be a simple*[2] *Lie algebra over an algebraically closed field of characteristic p with a Cartan subalgebra $\mathfrak{H}$. Suppose that $\mathfrak{L}$ has a nondegenerate representation form f. Then for h in $\mathfrak{H}$, $R_h^p=0$ implies $h=0$.*

PROOF. Let S be a representation of $\mathfrak{L}$ giving rise to f. We assume, without loss of generality, that S is irreducible, of degree m. By a result of Zassenhaus [1, p. 7], $\mathfrak{H}$ is abelian, so we can put the matrices $S_{\mathfrak{H}}$ in simultaneous triangular form. Now suppose that $h \in \mathfrak{H}$ and $R_h^p=0$. Then for any a in $\mathfrak{L}$,

$$[S_aS_h^p] = [\cdots \overbrace{[S_aS_h] \cdots S_h]}^{p} = S_{(\cdots \overbrace{(ah)\cdots h)}^{p}} = S_0 = 0.$$

Therefore by Schur's Lemma, S_h^p is a scalar matrix. Suppose the diagonal of S_h is $(\alpha_1, \cdots, \alpha_m)$. Then the diagonal of S_h^p is $(\alpha_1^p, \cdots, \alpha_m^p)$. Since S_k^p is a scalar, $\alpha_1^p = \cdots = \alpha_m^p$, so $\alpha_1 = \cdots = \alpha_m$. Now if k is any element of $\mathfrak{H}$, and the diagonal of k is $(\beta_1, \cdots, \beta_m)$ then the diagonal of S_hS_k is $(\alpha_1\beta_1, \cdots, \alpha_1\beta_m)$, so $\operatorname{tr}(S_hS_k)=\alpha_1 \operatorname{tr} S_k$. But $\operatorname{tr} S_k=0$ since $S_k \in [S_{\mathfrak{L}}S_{\mathfrak{L}}]$. Thus $f(h, k)=0$ for all k in $\mathfrak{H}$. But any invariant form which is nondegenerate on $\mathfrak{L}$ is nondegenerate on any Cartan subalgebra. Hence $h=0$ and the theorem is proved.

Now suppose that the hypotheses of Theorem 2 hold. Then $\mathfrak{L}$ must be centerless, and by Theorem 1, $\mathfrak{L}$ is restricted. Hence for any h in $\mathfrak{H}$, there is a unique h^p (necessarily in $\mathfrak{H}$) such that $R_h^p=R(h^p)$. The proofs of the diagonalizability of $\mathfrak{H}$ given in [1, p. 8] and [2, pp. 28, 29] now go through under the present weaker hypotheses. Briefly, the mapping $R_h \rightarrow R_h^p$ is a semilinear mapping of the space of diagonalizable R_k with k in $\mathfrak{H}$. This mapping is one-to-one by Theorem 2, and therefore, by a simple dimension argument, is onto. Hence every diagonalizable R_k, k in $\mathfrak{H}$, is the pth power of a diagonalizable R_h, h in $\mathfrak{H}$. But for any h in $\mathfrak{H}$, some power $R_h^{p^n}$ is diagonalizable, and so there is a diagonalizable R_k, k in $\mathfrak{H}$, with $R_k^{p^n}=R_h^{p^n}$. It follows from Theorem 2 that $R_k=R_h$. Hence the following result holds.

THEOREM 3. *Let $\mathfrak{L}$ be as in Theorem 2. Then for every h in $\mathfrak{H}$, R_h acts diagonally on $\mathfrak{L}$, that is, $xh=\alpha(h)x$ for every root α and every x in the root space $\mathfrak{L}_\alpha$.*

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[2] It is actually sufficient to assume that $\mathfrak{L}=\mathfrak{L}\mathfrak{L}$.

Theorems 2 and 3 generalize results of Jacobson (see [1, pp. 7–8]). It is possible to generalize similarly the results of Jacobson stated as Theorems 4.1 and 4.2 of [1]. Indeed, suppose that the hypotheses of Theorem 2 above are satisfied, and let U be a representation giving rise to the nondegenerate form f. We suppose without loss of generality that U is irreducible. Then for any x in $\mathfrak{L}$, by Schur's Lemma, $U_x^p - U(x^p)$ is a scalar matrix, and the trace of this scalar matrix is zero. Thus for any e_α and h,

$$\mathrm{tr}(U(e_\alpha)^p U_h) = \mathrm{tr}(U(e_\alpha^p) U_h),$$

and the proof of Theorem 4.1 given in [1, pp. 11–12] now goes through under the present hypotheses. Now if a nonzero root α and elements e_α in $\mathfrak{L}_\alpha$ and $e_{-\alpha}$ in $\mathfrak{L}_{-\alpha}$ are given then $e_\alpha^p = e_{-\alpha}^p = 0$, so $U(e_\alpha)^p$ and $U(e_{-\alpha})^p$ are scalar matrices of trace zero. Thus for a suitable linear functional λ on $\mathfrak{L}$, by adding the scalar matrix $\lambda(x)I$ to each U_x, we obtain a representation U' for which $U'(e_\alpha)^p = U'(e_{-\alpha})^p = 0$, and U' gives rise to the same trace form as U. With this change, the proof of Theorem 4.2 given in [1] is valid under the present hypotheses. Thus all roots are nonisotropic, that is, if α is a nonzero root and $f(h_\alpha, h) = \alpha(h)$ for all h in $\mathfrak{H}$, then $\alpha(h_\alpha) \neq 0$. For $p > 3$, this last result also follows from Theorem 3 by a theorem of Kaplansky [3, p. 165]. We summarize the results of this paragraph in the following theorem.

THEOREM 4. *Let $\mathfrak{L}$ be as in Theorem 2 and let α be a nonzero root. Then α is nonisotropic, and if $e_\alpha \in \mathfrak{L}_\alpha$, $e_\alpha^p = 0$.*

As noted in [3], Seligman makes no further use of the fact that his trace form arises from a restricted representation. Thus the result stated in our introduction holds, that is, Seligman's Theorem 16.2 [1, p. 77] remains valid without the hypothesis of restrictedness of $\mathfrak{L}$ and its representation.

In particular it follows that the invariant forms of the algebras $\mathfrak{S}_3$ and $\mathfrak{L}(\mathfrak{G}, \delta, f)$ given in [4] do not arise from a representation.

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