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SARIO'S LEMMA ON HARMONIC FUNCTIONS¹

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1. The alternating method of constructing a harmonic function on a Riemann surface due to H. A. Schwarz is discussed from a general point of view by Sario [2]. To show the convergence of the Neumann series, he uses the following lemma ([2, p. 282]; it has another expression but is evidently equivalent to the following):

SARIO'S LEMMA. *Let W be an open Riemann surface and K be a compact set on W . Then there exists a constant q such that $0 < q < 1$ and that*

$$\max_K |u| \leq q \cdot \sup_W |u|$$

for every harmonic function u which changes sign on K .

The present note is devoted to evaluating the constant q according to oral suggestions of Professor S. Warschawski, to whom the author wishes to express his heartiest gratitude. We are of course interested in finding the smallest possible value. However, we have succeeded in doing so only for a very special case.

2. It is well known that we can introduce the Poincaré metric

$$ds = \frac{|dz|}{1 - |z|^2}$$

in the unit disc $|z| < 1$. It is invariant under linear transformations of $|z| < 1$ onto itself.

For z_1 and z_2 , the infimum of $\int_C ds$ with respect to all the curves C connecting z_1 with z_2 will be denoted by $d(z_1, z_2)$. For instance,

$$(1) \quad d(0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}.$$

The geodesic connecting z_1 with z_2 is the arc contained in the circle perpendicular to $|z| = 1$ (see, e.g., [1]).

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Let E be a set contained in $|z| < 1$. The supremum of $d(z_1, z_2)$ with respect to every pair of $z_1, z_2 \in E$ will be called the *Poincaré diameter* of E . If E is compact the supremum is the maximum.

3. THEOREM 1. *If W is the unit disc $|z| < 1$, then*

$$\frac{2}{\pi} \operatorname{Arctan} \sinh 2D,$$

where D is the Poincaré diameter of K , is the smallest possible value for q in Sario's lemma. Here Arctan is the principal value of \arctan .

For the proof, we use the following lemma by the suggestion of Professor S. Warschawski:

LEMMA. *Let $u(z)$ be a harmonic function in $|z| < 1$ such that $|u(z)| \leq 1$ and $u(0) = 0$. Then*

$$|u(z)| \leq \frac{2}{\pi} \operatorname{Arc} \tan \frac{1}{1-r^2} \quad \text{in } |z| \leq r < 1.$$

The equality sign is realized by

$$u_0(z) = \operatorname{Re} \frac{2i}{\pi} \operatorname{Log} \frac{1+z}{1-z}$$

at $z = -ir$, where Log is the principal value of \log .

PROOF OF THE LEMMA. Let $w(z) = u(z) + iv(z)$ where v is the conjugate of u such that $v(0) = 0$. Let $W = S(w)$ be the inverse function of $w = (2i/\pi) \operatorname{Log} ((1+W)/(1-W))$. $S(w)$ maps $|\operatorname{Re} w| < 1$ conformally onto $|W| < 1$ and is such that $S(0) = 0$. The lemma is proved immediately when we apply Schwarz's lemma to $S \circ w(z)$. We omit the proof in detail.

To prove Theorem 1, it is evidently sufficient to prove the following proposition:

Let $u(z)$ be a function such that

- (i) harmonic in $|z| < 1$;
- (ii) $|u(z)| \leq 1$ in $|z| < 1$;
- (iii) changes sign on the set K which is compact in $|z| < 1$. Then

$$|u(z)| \leq \frac{2}{\pi} \operatorname{Arctan} \sinh 2D \quad \text{on } K,$$

where D is the Poincaré diameter of K . This estimate is the best possible.

PROOF. Take points $z_1, z_2 \in K$ such that $u(z_1) > 0$ and $u(z_2) < 0$. Let g be the geodesic connecting z_1 with z_2 . There exists a $z_0 \in g$ such that $u(z_0) = 0$. z_0 is not necessarily contained in K , but we have

$$(2) \quad d(z, z_0) < D \quad \text{for any } z \in K.$$

In fact, bring z to 0 by a linear transformation. The images of z_j ($j=0, 1, 2$) under the transformation are denoted by z'_j ($j=0, 1, 2$), respectively. The geodesic g is transformed to an arc g' on a circle perpendicular to $|z|=1$. Since g' connects z'_1 with z'_2 and $z_0 \in g'$, we get immediately that $|z'_0| < \max(|z'_1|, |z'_2|)$. Thus we have $d(z, z_0) < \max(d(z, z_1), d(z, z_2)) \leq D$ because $|z'_j| = d(0, z'_j) = d(z, z_j)$ ($j=0, 1, 2$). Let

$$Z = T(z) = \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Then, by (2), the images of K under T is contained in the circle $|Z| < r$, where r is determined by $D = (1/2) \log ((1+r)/(1-r))$ (see (1)). The previous lemma is applicable to the function $u(T^{-1}(Z))$ and we get

$$\begin{aligned} \max_K |u(z)| &\leq \max_{|Z| \leq r} |u(T^{-1}(Z))| \\ &\leq \frac{2}{\pi} \text{Arctan} \frac{1}{1 - r^2} = \frac{2}{\pi} \text{Arctan} \sinh 2D. \end{aligned}$$

To show that this estimate is the best possible, take points $z_1, z_2 \in K$ such that $d(z_1, z_2) = D$. For an ϵ ($0 < \epsilon < D$), take the point $z_0 \in g$ such that $d(z_1, z_0) = \epsilon$; here g is the geodesic connecting z_1 with z_2 . Let

$$Z = T_\epsilon(z) = e^{i\alpha} \frac{z - z_0}{1 - \bar{z}_0 z}$$

where α is chosen so that $T_\epsilon(z_0)$ is contained in the positive imaginary axis. By making use of the function $u_0(Z)$ introduced in the lemma, put

$$u_\epsilon(z) = u_0(T_\epsilon(z)).$$

It is not difficult to see that $u_\epsilon(z_1) < 0$ and

$$u_\epsilon(z_2) = \frac{2}{\pi} \text{Arctan} \frac{2 |T_\epsilon(z_2)|}{1 - |T_\epsilon(z_2)|^2}.$$

From the relation $d(0, T(z_2)) = d(z_0, z_2) = D - \epsilon$, we obtain, by (1), that

$$|T_\epsilon(z_2)| = \frac{e^{2(D-\epsilon)} - 1}{e^{2(D-\epsilon)} + 1}$$

and, therefore,

$$u_\epsilon(z_2) = \frac{2}{\pi} \text{Arctan} \sinh 2(D - \epsilon).$$

Since ϵ is arbitrarily small, it is shown that our estimate is the best possible.

4. Let W be a hyperbolic Riemann surface, i.e., its universal covering surface is $|z| < 1$. The metric

$$ds = \frac{|dz|}{1 - |z|^2},$$

invariant under cover transformations, can be projected onto W . The projected metric is determined independently of the choice of the projection. In terms of this projected metric, we define analogously the Poincaré distance $d(p_1, p_2)$ between the points $p_1, p_2 \in W$ and the *Poincaré diameter* of a set $E \subset W$.

THEOREM 2. For a hyperbolic open Riemann surface W ,

$$\frac{2}{\pi} \operatorname{Arctan} \sinh 2D,$$

where D is the Poincaré diameter of K , can be a value of q in Sario's lemma.

The author does not know if it is the smallest possible for W which is hyperbolic and carries a nonconstant bounded harmonic function.

PROOF. Again we may estimate only a function $u(p)$ such that

- (i) harmonic on W ;
- (ii) $|u(p)| \leq 1$ on W ;
- (iii) changes sign on K .

Take $p_1, p_2 \in K$ such that $u(p_1) > 0$, $u(p_2) < 0$. Let $|z| < 1$ be the universal covering surface of W such that the point $z=0$ lies over p_1 . Among all the points in $|z| < 1$ lying over p_2 , take the one, z_2 , which is nearest to the origin (in either metric), as follows from the definition of the universal covering surface. It satisfies

$$(3) \quad |z_2| = d(0, z_2) = d(p_1, p_2) < D.$$

We can lift u up and regard this as a function defined in $|z| < 1$. u changes sign on the set $\{0, z_2\}$. By (3), the Poincaré diameter of $\{0, z_2\}$ does not exceed D . Therefore, on applying Theorem 1 to $u(z)$ regarding the set $\{0, z_2\}$ as K , we obtain the desired estimate.

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