A GENERALIZATION OF THE CARTAN-BRAUER-HUA THEOREM

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Let K be a division ring, K' its multiplicative group, and M a subdivision ring. In a recent note, Schenkman [3] proved that if M is a subfield whose multiplicative group M' is subinvariant in K', then either $M=K$ or M is contained in the center of K. For the case of noncommutative M, the same result was obtained provided M contains at least five elements of the center of K. However, there is a gap in the proof for the noncommutative case.[1] It is the purpose of this paper to fill this gap and to remove the restriction on M in the noncommutative case. Combining the result here with the result in [3], we will have

THEOREM. *The only subinvariant subdivision rings of a division ring K are K itself and subfields of the center of K.*

If S is a subset of K, then $Z(S)$ will denote its centralizer in K and $N(S)$ its normalizer in K'. For brevity, $N(N(S))$ will be written $N^2(S)$. The notation $J\triangle L$ means that J is an invariant subgroup of L. A subgroup or subdivision ring M_1 is a conjugate of M *via* H if there is an $x\in H$ such that $M_1=M^x=x^{-1}Mx$. If A and B are sets, then $A<B$ means that A is a proper subset of B, while $B\backslash A$ denotes the complement of A in B.

For convenience, Lemmas 1 and 2 of [3] will be restated here.

LEMMA 1. *If M is a subdivision ring of a division ring K, $x\in N(M)$, $x\notin M$, $x\notin Z(M)$, $m\in Z(M)\cap M'$, then $m+x\notin N(M)$.*

LEMMA 2. *If H and M are subdivision rings of K, $H\not\subset M$, $H'\subset N(M)$, then $H\subset Z(M)$.*

LEMMA 3.[2] *Let L and M be subdivision rings of K with L noncommutative. Then the index $[L' : L'\cap N(M)]\neq 2$.*

PROOF. Let $P=L\cap M$, $Q=L\cap Z(M)$, $R=L'\cap N(M)$. Then $P'\triangle R$ and $Q'\triangle R$. Now deny the lemma. Assume that $x\in R$, $x\notin M$, $x\notin Z(M)$. By Lemma 1, $x+1\notin N(M)$ and $x-1\notin N(M)$. Since

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[1] The gap occurs in the first half of the next to the last sentence in the paper, where the case $N(M)=N^2(M)$ is not considered.

[2] For closely related results, see Faith [1, Theorem 3 and Corollary 1].

$[L':R]=2$, $x^2-1=(x+1)(x-1)\in R$. Since $x^2\in N(M)$, it follows from Lemma 1 that $x^2\in M$ or $x^2\in Z(M)$, and so $x^2\in P$ or $x^2\in Q$. Thus for any $y\in R$, $y^2\in P$ or $y^2\in Q$. It follows that

(1) If $u\in L$, then $u^4\in P$ or $u^4\in Q$.

We assert that

(2) There is a function f from L to the natural numbers such that either (i) $u^{f(u)}\in P$ for all $u\in L$, or (ii) $u^{f(u)}\in Q$ for all $u\in L$.

In fact, if (2) is false, there are elements u and v of L such that $u^n\notin Q$, $v^n\notin P$ for all natural numbers n. But then by (1), $u^4\in P$, $v^4\in Q$, and by the definitions of P and Q, u^4 and v^4 permute. By (1), $(u^4v^4)^4\in P$ or Q. Hence $v^{16}\in P$ or $u^{16}\in Q$, a contradiction. Hence (2) holds.

In either case of (2), there is a proper subdivision ring S (one of the rings P or Q) of L such that $u^{f(u)}\in S$ for all $u\in L$, and such that S has just one or two conjugates via L, namely S and (perhaps) S^*. Clearly there is a function g from L to the natural numbers such that $u^{g(u)}\in S^*$ for all $u\in L$. Since $(S\cap S^*)'\triangle L'$, by the Cartan-Brauer-Hua theorem, $S\cap S^*\subset Z(L)$. Therefore $u^{f(u)g(u)}\in Z(L)$ for all $u\in L$. By [2], L is commutative, a contradiction.

LEMMA 4. *If K is a division ring and M a noncommutative proper subdivision ring, then M' is not subinvariant in K'.*

PROOF. Deny the lemma. Let n be the smallest natural number such that there exist a division ring K, a noncommutative subdivision ring M and subgroups $G_1, \cdots, G_n$ such that $M'\triangle G_1\triangle G_2\triangle \cdots \triangle G_n=K'$.

If $x\in K'$ and $M<M^x$, then $M'\triangle(G_1\cap M^x)\triangle \cdots \triangle(G_{n-1}\cap M^x)=(M^x)'$, contradicting the minimality property of n. Hence $M\not< M^x$ if $x\in K'$.

By Lemma 2, if $x\in G_2$ and $M^x\neq M$, then $M\subset Z(M^x)$. By the Cartan-Brauer-Hua theorem, $n\neq 1$. Hence by the minimality property of n, there is a conjugate $M^y\neq M$ of M with $y\in G_2$. Let Q be the intersection of all centralizers of conjugates of M via G_2, other than M and M^y, and let $Q=K$ if there are no such conjugates. Then $M\subset Q$ and $M^y\subset Q$. It is then clear that M has at most two conjugates via $G_2\cap Q$. Now $M'\triangle G_1\cap Q\triangle \cdots \triangle G_n\cap Q=Q'$. Hence, returning to the original notation, it may be assumed that M has just two conjugates via G_2, namely M and M^y.

If $M<Z(M^y)$, then M is the only conjugate of itself via $G_2\cap Z(M^y)$, and the minimality of n is contradicted in the usual way. Hence $M=Z(M^y)$, and similarly $M^y=Z(M)$. If $x\in N(M)$, then $M^{yx}=(Z(M))^x=Z(M^x)=Z(M)=M^y$, so that $N(M^y)\supset N(M)$. Similarly $N(M)$

$\supset N(M^y)$. Hence $N(M^y)=N(M)$. Therefore $N(M)^y=N(M^y)=N(M)$, and $y\in N^2(M)\backslash N(M)$.

If $x\in G_2$, then $M^x=M$ or M^y, so that $x\in N(M)$ or $x\in N(M)y$. Hence $G_2\subset N^2(M)$. Now if $G_2=K'$, then M has just two conjugates in K, in contradiction to Lemma 3. Thus $G_2<K'$. If $G_3\subset N^2(M)$, then $M'\triangle G_3\cap N(M)\triangle G_3\cap N^2(M)=G_3$, contradicting the minimality of n. Hence there is a $u\in G_3\backslash N^2(M)$. Then $(M^u)'\subset G_2\subset N^2(M)$. If $M^u \not\subset N(M)$, then $[(M^u)': M^u \cap N(M)] = 2$. This contradicts Lemma 3. Therefore $(M^u)'\subset N(M)$. Since $u\notin N^2(M)$, $M^u\neq M$ or M^y. By Lemma 2, $M^u<Z(M)=M^y$. Thus $M<M^{yu^{-1}}$, which was shown to be impossible in the first part of the proof.

Bibliography

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