$1-1+1-1+\cdots = \frac{1}{2}$

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It is of course well known that $1-1+1-1\cdots$ is Abel-summable to $1/2$, that is to say

$$1 - t + t^2 - t^3 + \cdots \to \frac{1}{2} \qquad \text{as } t \to 1^-.$$

It can also be shown that

$$1 - t + t^4 - \cdots \pm t^{n} \cdots \to \frac{1}{2} \qquad \text{as } t \to 1^-.$$

If we consider, however, the rate at which these two functions approach their limit $(1/2)$ then we find that these are worlds apart! In fact

$$(1 - t + t^2 \cdots) - \frac{1}{2} = \frac{1}{1+t} - \frac{1}{2} = \frac{1-t}{2(1+t)} \sim \frac{1-t}{4},$$

whereas an application of the functional equation for the θ-function gives

$$(1 - t + t^4 - \cdots) - \frac{1}{2} \sim \left(\frac{\pi e^{-\pi}}{1-t}\right)^{1/2} \exp\left(-\frac{\pi^2}{4}\cdot\frac{1}{1-t}\right).$$

In this note we show that this anomalous behavior persists for the other functions

$$(1) \qquad f_k(t) = \sum_{n=0}^{\infty} (-1)^n t^{n^k}, \qquad k = 1, 2, \cdots,$$

that namely $f_k(t)$ approaches $1/2$ very slowly for k odd and very quickly for k even.

THEOREM. *If $f_k(t)$ are defined by* (1) *then*
A. *for any fixed odd k there is a $c>0$ such that* $|f_k(t)-1/2| > c(1-t)$,
B. *for any fixed even k there is a $c>0$ such that*

$$|f_k(t) - 1/2| < A \exp(-c/(1-t)^{\alpha}), \qquad \alpha = 1/(k-1).$$

PROOF. Our principal tool will be the Mellin inversion formula which gives

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$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)x^{-s}ds = e^{-x}, \qquad x > 0.$$

From this we obtain, by a term by term integration,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)x^{-s}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{ks}} = \sum_{n=1}^{\infty}(-1)^{n-1}e^{-nkx}$$
$$= 1 - f_k(e^{-x}).$$

Now note that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{ks}} = \left(1-\frac{2}{2^{ks}}\right)\sum_{n=1}^{\infty}\frac{1}{n^{ks}} = (1-2^{1-ks})\zeta(ks)$$

and obtain

$$\text{(2)} \qquad \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)x^{-s}(1-2^{1-ks})\zeta(ks)ds = 1 - f_k(e^{-x}).$$

We now shift the contour from the vertical line Re $s=2$ to the line Re $s=-1/k$. (It is assumed that $k>1$.) The estimates

$$|\Gamma(s)| < Ae^{-c|t|}, \qquad |\zeta(ks)| < |t|^A, \qquad t = \operatorname{Im} s, \ A, c > 0,$$

are known to be valid in $-1/k \leqq \operatorname{Re} s \leqq 2$, $|t|>1$ and so the Residue theorem is applicable.

However, in $-1/k \leqq \operatorname{Re} s \leqq 2$, the only pole is seen to be at $s=0$ with residue $(1-2)\zeta(0)=1/2$ and so we obtain

$$\text{(3)} \qquad \frac{1}{2\pi i}\int_{-1/k-i\infty}^{-1/k+i\infty}\Gamma(s)x^{-s}(1-2^{1-ks})\zeta(ks)ds = \frac{1}{2} - f_k(e^{-x}).$$

Let us now utilize the functional equation of the ζ-function, namely

$$\text{(4)} \qquad \zeta(s) = \frac{1}{\pi}(2\pi)^s \sin\frac{\pi}{2}s\Gamma(1-s)\zeta(1-s).$$

If we introduce this into (3) and then call $1-ks=z$ the result is

$$\text{(5)} \qquad -\frac{1}{k\pi i}\int_{2-i\infty}^{2+i\infty}\pi^{-z}\Gamma\left(\frac{1-z}{k}\right)\Gamma(z)\cos\frac{\pi}{2}zx^{(z-1)/k}(1-2^{-z})\zeta(z)dz$$
$$= \frac{1}{2} - f_k(e^{-x}).$$

PROOF OF A. Here k is odd. If we shift the contour from 2 to $2k$ then the residue theorem is again applicable and it gives one residue at $z=k+1$, namely cx where

$$c = \frac{2(k)!}{\pi^{k+1}}(1 - 2^{-(k+1)})\zeta(k+1)(-1)^{(k-1)/2}.$$

The remaining contour integral is the same as that in (5) with the limits changed to $2k-i\infty$, $2k+i\infty$. This clearly has magnitude

$$\leqq x^{(2k-1)/k}\int_{2k-i\infty}^{2k+i\infty} M\cdot\left|\Gamma\left(\frac{1-z}{k}\right)\right|\,|dz| = cx^{(2k-1)/k}$$

and so finally, by (5)

$$\frac{1}{2} - f_k(e^{-x}) = cx + O(x^{2-1/k}), \qquad k > 1,$$

which yields A immediately.

PROOF OF B. Here k is even and so the integrand in (5) is analytic for all z, Re $z>0$. We may then shift the contour to the right to the line Re $z=kM+1$ ($M=$half an odd integer) and no residues will be introduced.

We now estimate the resulting integral. We have

$$\left|\pi^{-z}x^{(z-1)/k}(1-2^{-z})\zeta(z)\right| < x^M,$$

$$\left|\Gamma(z)\cos\frac{\pi}{2}z\right| \leqq (kM)!,$$

$$\left|\Gamma\left(\frac{1-z}{k}\right)\right| = \left|\Gamma\left(-M+\frac{iy}{k}\right)\right|, \qquad (y = \operatorname{Im} z),$$

$$\leqq \left|\Gamma\left(+\frac{1}{2}-\frac{iy}{k}\right)\right|\cdot\frac{1}{\left(M-\frac{1}{2}\right)!}$$

$$= \left[\frac{\pi}{\cosh\frac{y}{k}}\right]^{1/2}\cdot\frac{1}{\left(M-\frac{1}{2}\right)!}$$

and so the resulting integral is in magnitude

$$\leqq c_1\frac{(kM)!}{\left(M-\frac{1}{2}\right)!}x^M \leqq c_2M^{(k-1)M}x^M.$$

If M is now chosen close to $x^{-\alpha}/e$, $\alpha=1/k-1$ then the resulting estimate becomes

$$< Ae^{-c/x^\alpha}$$

and B follows immediately. This completes the proof.

These results seem to point to the fact that of all the series $\sum(-1)^n t^{f(n)}$ the one which goes "fastest" to 1/2 is $\sum(-1)^n t^{n^2}$. I wish to thank J. Korevaar for pointing out that this is actually the case, in the following sense:

We have seen that $\left|\sum(-1)^n t^{n^2}-1/2\right| \leqq e^{-c/(1-t)}$. On the other hand, it is impossible that

$$\left|\sum (-1)^n t^{f(n)} - \frac{1}{2}\right| \leqq e^{-\phi(t)/(1-t)}$$

where $\phi(t) \geqq 0$ is unbounded as $t \to 1$. It is, in fact, a theorem of Korevaar [3], that if

$$\left| \sum A_n e^{-nu} \right| \leqq w(u), \qquad u > 0,$$

and

$$A_n \geqq -1, \qquad \liminf_{u\to 0^+} u \log w(u) = -\infty,$$

then $\sum A_n e^{-nu} = \text{constant}$.

Changing u into $\log(1/t)$ immediately yields the result stated above.

Bibliography

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