ON RINGS WITH ONE-SIDED FIELD OF QUOTIENTS

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This note extends certain results of Cartan-Eilenberg's *Homological algebra*, Chapter VII, *Integral domains*, to rings with one-sided field of quotients. Let R be a ring with a left field of quotients Q. We prove the following results. If A is a left R-module and tA the set of all its torsion elements then $\operatorname{Tor}_1^R(Q/R, A) = tA$. With a slightly modified definition of inversible ideal it is proved that the notions of projective and inversible ideals are equivalent. If, in addition, R is left semi-hereditary then every finitely generated torsion-free right R-module is R-flat. Every finitely generated torsion-free left R-module can be imbedded in a finitely generated free left R-module if and only if R has both left and right field of quotients. Finally we give an example of a ring having a left field of quotients but not a right one. This implies that there exist finitely generated torsion-free left R-modules which cannot be imbedded in a projective left R-module.

1. **Generalities.** Our basic reference will be [1]. Let R be a ring with identity and without zero divisors. R is said to have a left field of quotients (see [**2**, Chapter I, §9, Exercises 8 and 9]) if R is a subring of a division ring Q in which every element is a "left quotient" of elements in R, that is, every element in Q can be written as $x^{-1}\cdot y$, where $x\in R$, $y\in R$ and $x\neq 0$. The following conditions for R to be a ring with a left field of quotients are equivalent:

G1. For every pair $(x, x')\in R\times R$, $x'\neq 0$, there exists a pair $(u, v) \in R\times R$, $u\neq 0$ such that $u\cdot x = v\cdot x'$.

G2. The intersection of any two nonzero left ideals of R is different from zero.

G3. There exists a division ring Q containing R such that for every finite set $\{q_1, \cdots, q_n\}$ of elements in Q, there exists an element $r\in R$, $r\neq 0$ such that $r\cdot q_i\in R$ for $i=1, \cdots, n$.

As in [1], in any module A we can consider the set of torsion elements tA and the set of divisible elements δA. If R has a left field of quotients and A is a left (resp. right) R-module then tA (resp. δA) is a submodule of A. This follows immediately from G1.

PROPOSITION 1.1. *A torsion-free left R-module is injective if and only if it is divisible.*

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PROOF. The necessity is identical to the necessity of Proposition 1.2, [1, Chapter VII]. To prove the sufficiency assume that A is a torsion-free and divisible left R-module. Consider a homomorphism $f\colon I \to A$ of a left ideal I of R into A. For every $i \in I$, $i \neq 0$ there exists a unique $a_i \in A$ such that $fi = i \cdot a_i$. Let j be any nonzero element of I. Let u and v, $u \neq 0$ be elements of R satisfying $u \cdot i = v \cdot j$. Then we have

$$u \cdot i \cdot a_i = u \cdot fi = f(u \cdot i) = f(v \cdot j) = v \cdot fj = v \cdot j \cdot a_j = u \cdot i \cdot a_j$$

and therefore $a_i = a_j = a$. Then for every $j \in I$, $fj = j \cdot a$ and therefore A is injective.

2. **The field of quotients.** Let Q be the left field of quotients of R. We introduce in R the following partial order. (1) For every x, $0 < x$, (2) if $x \neq 0$ and $y \neq 0$ $x < y$ if and only if $x^{-1} \cdot R \subset y^{-1} \cdot R$. $[R, <]$ is a directed set. In fact, let x and y be any two nonzero elements of R. There exist two elements u, v in R $u \neq 0$ and $v \neq 0$ such that $u \cdot x = v \cdot y$. Then

$$x^{-1} \cdot R \subset x^{-1} \cdot u^{-1} \cdot R = (u \cdot x)^{-1} \cdot R \quad \text{and} \quad y^{-1} \cdot R \subset y^{-1} \cdot v^{-1} \cdot R = (v \cdot y)^{-1} \cdot R$$

and thus $x < u \cdot x$ and $y < u \cdot x$.

Let $R_x = x^{-1} \cdot R$ if $x \neq 0$ and $R_0 = 0$. We define $f_{xy}\colon R_x \to R_y$ for every pair $x < y$, by inclusion. Then $\{R_x, f_{yz}\}$ becomes a directed system of right R-modules and it is obvious that the direct limit is Q. Since each R_x is R-projective we have

PROPOSITION 2.1. *Q is a right flat R-module.*

From the exact sequence $0 \to R \to Q \to Q/R \to 0$ and for every left R-module A we have the exact sequence

$$0 \to \operatorname{Tor}_1(Q/R, A) \to A \to Q \otimes A \to Q/R \otimes A \to 0 \qquad (\otimes = \otimes_R).$$

Now by using G3 one can prove

PROPOSITION 2.2. $\operatorname{Tor}_1^R(Q/R, A) = tA$.

PROOF. The same as in the commutative case. See [2, Chapter III §2, Theorem 2 and also Exercise §2–7g].

Then if A is a torsion-free left R-module we have the exact sequence

$$0 \to A \to Q \otimes A \to Q/R \otimes A \to 0. \tag{1}$$

Observe that since $Q \otimes A$ is torsion-free and divisible it is R-injective. Applying cohomology to (1) we get the isomorphism

$$\operatorname{Hom}_R(A', K \otimes A) \approx \operatorname{Ext}_R^1(A', A) \qquad (K = Q/R), \tag{2}$$

where A' is a torsion module (i.e. $tA' = A'$) and A is torsion-free.

Now we recall

G4. R has a left field of quotients if and only if for every nonzero left ideal I of R, R/I is a torsion module.

Then G4 and (2) give the isomorphism

$$\mathrm{Hom}_R(R/I, K) \approx \mathrm{Ext}^1_R(R/I, R). \tag{3}$$

$\mathrm{Ext}^n_R(R/I, K)$, $n>1$ can be computed by using the exact sequences

$$0 \to I \to R \to R/I \to 0,$$
$$0 \to R \to Q \to K \to 0,$$

the fact that Q is R-injective, and Theorem 1.5 [1, Chapter VI].

3. **Inversible ideals.** A left ideal I of R is called *inversible* if there exist elements $q_1, \cdots, q_n$ in Q and $a_1, \cdots, a_n$ in I such that $I \cdot q_i \subset R$ and $\sum_i q_i \cdot a_i = 1$, $i = 1, \cdots, n$.

PROPOSITION 3.1. *In order that a nonzero ideal I of R be projective it is necessary and sufficient that I be an inversible ideal.*

PROOF. The sufficiency is the same as Proposition 3.2 [1, Chapter VII]. Let I be a projective left ideal and let $\{a_i\}$, $\{\phi_i\}$ be families of elements of I and R-maps $\phi_i\colon I \to R$ respectively, satisfying $a = \sum_i \phi_i(a) \cdot a_i$ for every $a \in I$. Let $x \neq 0$, $x \in I$. We define $q_i \in Q$ by $q_i = x^{-1} \cdot \phi_i(x)$. Let y be any element of I. There exist u and v in R, $v \neq 0$ such that $u \cdot x = v \cdot y$. Then $v \cdot \phi_i(y) = \phi_i(v \cdot y) = \phi_i(u \cdot x) = u \cdot \phi_i(x) = u \cdot x \cdot q_i = v \cdot y \cdot q_i$ and therefore $\phi_i(y) = y \cdot q_i$ for every $y \in I$ and furthermore $I \cdot q_i \subset R$ for every i.

Since $\phi_i(x) = x \cdot q_i$ it follows that $q_i = 0$ for almost every i. Finally

$$x = \sum_i (\phi_i x) \cdot a_i = \sum_i x \cdot q_i \cdot a_i = x \sum_i q_i \cdot a_i \quad \text{implies} \quad \sum_i q_i \cdot a_i = 1.$$

By using the same proof as Proposition 3.3 and Proposition 3.4 [1, Chapter VII] we have

COROLLARY 3.1. *Any inversible ideal is finitely generated.*

COROLLARY 3.2. *If I is inversible and A divisible then* $\mathrm{Ext}^1_R(R/I, A) = 0$.

This last corollary implies

COROLLARY 3.3. *If, in addition, R is left hereditary a left R-module is divisible if and only if it is injective.*

By using Corollary 3.1 we can prove

PROPOSITION 3.2. *Let R be a left hereditary ring. Then every torsion-free right R-module C is flat.*

PROOF. By Corollary 3.1 R is noetherian. Then using the Proposition 5.5 [1, Chapter VI] we have the monomorphism

$$\mathrm{Tor}_1^R(C, R/I) \to \mathrm{Hom}_R(\mathrm{Ext}_R^1(R/I, R), C).$$

(3) in §2 gives the monomorphism

$$\mathrm{Tor}_1^R(C, R/I) \to \mathrm{Hom}_R(\mathrm{Hom}_R(R/I, K), C) \tag{1}$$

where C is a torsion-free right R-module. Since $\mathrm{Hom}(R/I, K)$ is a torsion module the second part of (1) is zero and hence $\mathrm{Tor}_1^R(C, R/I) = 0$ for every I, which means that C is right R-flat.

NOTE. It can be seen that the monomorphism of Proposition 5.5, [1, Chapter VI] also holds assuming that R is left semihereditary and A is finitely generated and with a projective resolution composed of finitely generated modules. Since this is the case if $A = R/I$ for I finitely generated, it follows by using the argument of the last proposition that $\mathrm{Tor}_1^R(C, R/I) = 0$ for I finitely generated. Now, by taking direct limit this holds for every left ideal I. Then our last proposition can be stated for left semihereditary rings and this gives the generalization of Proposition 4.2 [1, Chapter VII].

4. Imbedding in a free module. We denote by (F) the following property:

(F) Every finitely generated torsion-free left R-module can be imbedded in a finitely generated free left R-module.

PROPOSITION 4.1. *R satisfies* (F) *if and only if in addition R has a right field of quotients.*

PROOF. If R satisfies (F) let x and y be any two nonzero elements of R. Let $A = R\cdot x^{-1} + R\cdot y^{-1}$ be the left R-submodule of Q generated by x^{-1} and y^{-1}. Let F be a finitely generated free left R-module containing A. Let $f_1, \cdots, f_n$ be a basis of F. Then A (and a fortiori F) contains an element $e = x\cdot x^{-1} = y \cdot y^{-1}$. Let $x^{-1} = \sum_i r_i \cdot f_i$ and $y^{-1} = \sum_i s_i \cdot f_i$ with $r_i \in R$, $s_i \in R$ and $r_1 \neq 0$ and $s_1 \neq 0$. Then $e = \sum_i x\cdot r_i \cdot f_i = \sum_i y \cdot s_i \cdot f_i$ and this implies $x\cdot r_1 = y \cdot s_1$ which means that that R has a right field of quotients.

If R has a right field of quotients let A be a finitely generated torsion free left R-module. Then A can be imbedded in $Q \otimes A$, which is a left Q-vector space. If $a_1, \cdots, a_n$ generate A then $1 \otimes a_1, \cdots, 1 \otimes a_n$ generate $Q \otimes A$ as a Q-vector space. Let $e_1, \cdots, e_m$ be a basis of $Q \otimes A$. Then for every $i = 1, \cdots, n$

$$a_i = \sum_j q_{ij} \cdot e_j, \qquad q_{ij} \in Q.$$

By the dual of G3 there exists $r \in R$, $r \neq 0$ such that $q_{ij} \cdot r \in R$ for every pair ij. Then $a_i = \sum_j (q_{ij}r)(r^{-1}\cdot e_j)$ and therefore A is contained in the free left R-module generated by $r^{-1}\cdot e_1, \cdots, r^{-1}\cdot e_m$.

In the next section we will give an example of a ring having a left field of quotients but not a right one. This example and Proposition 4.1 imply that there exist torsion-free finitely generated left modules which cannot be imbedded in a projective left R-module.

It is easy to see that every ring without zero divisors in which every left ideal is principal is a ring with a left field of quotients. Then if R is a left and right principal ideal ring every finitely generated torsion-free left (or right) R-module is free.

PROPOSITION 4.2. *Let R be a left semihereditary ring with a left field of quotients. Then R satisfies the condition* (F) *stated for right ideals.*

PROOF. Let A be a finitely generated torsion free right R-module. Then A is R-flat, therefore $\mathrm{Tor}_1^R(A, Q/R) = 0$ and A can be imbedded in $A \otimes Q$. Now the proof is the same as the second part of Proposition 4.1.

5. **An example.** We will give in this section an example of a ring having a left field of quotients but not a right one. Let K be a commutative field and $f: K \to K'$ an isomorphism of K onto a subfield K' of K satisfying $[K: K'] > 1$. Let $R = K[X, f]$ be the ring of (twisted) polynomials $\sum r_i \cdot X^i$ with the ordinary sum and with the following product:

$$(r\cdot X^i) \circ (s \cdot X^j) = r \cdot f^i(s) \cdot X^{i+j}.$$

It is easy to see that $[R, \circ]$ is a ring (see [2, Chapter IV, Exercise, §5, 10]). The verification that R satisfies G1 is immediate and then R has a left field of quotients. In order to prove that R has no right field of quotients we must observe that if E is any K'-vector space contained in K the set (par abus de langage) denoted by $E[X]$ of all the polynomials of R with coefficients in E is a right ideal of R. Since $[K: K'] > 1$ we can find two nonzero vector spaces E and E' over K' satisfying $E \cap E' = 0$. But this implies $E[X] \cap E'[X] = 0$ and so R does not satisfy the dual of G2 and therefore has not a right field of quotients.

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