A NOTE ON METRIC DENSITY OF SETS OF REAL NUMBERS

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Casper Goffman [1] has shown that the set of points at which the metric density of a set of real numbers exists but is not zero or one is a set of the first category. As a partial converse to this result he showed that for every F_σ set of measure zero there exists a measurable set whose density exists at every point of the F_σ and has the value 1/2. In this note we extend the last named theorem of Goffman to the following:

THEOREM. *Let Z and γ be given where Z is an F_σ set of measure zero and γ is a real number such that $0<\gamma<1$. Then there exists a measurable set S such that the metric density of S exists at every point of Z and has the value γ.*

PROOF. We shall assume that $Z\subset(0, 1)$. Let $Z=\bigcup_{k=1}^{\infty} Z_k$ where Z_k is closed and of measure zero for $k=1, 2, \cdots$.

We shall define four sequences, $\{G_k\}$, $\{T_k\}$, $\{E_k\}$, and $\{F_k\}$ of sets, where $G_{k+1}\subset G_k$ and $T_{k+1}\subset T_k$, as follows:

Let $G_1=(0, 1)$, and G_k be an open set which contains $Z-\bigcup_{n=1}^{k-1} Z_n$. Define T_k to be the set G_k-Z_k and require $G_{k+1}\subset T_k$. Since Z_k is closed, T_k is open and consists of a countable number of disjoint open intervals $I_{kj}=(a_{kj}, b_{kj})$. Since $m(I_{kj})<1$, where m denotes Lebesgue measure, there exists an integer N_{kj} such that

$$\frac{1}{N_{kj}+1} \leqq \frac{1}{2} m(I_{kj}) \leqq \frac{1}{N_{kj}} \cdot$$

Let $\alpha_n^{kj}=a_{kj}+(1/2)m(I_{kj})=b_{kj}-(1/2)m(I_{kj})=\beta_n^{kj}$ for $n=N_{kj}$ and for $n\geqq N_{kj}+1$ let

$$\alpha_n^{kj} = a_{kj} + \frac{1}{n}, \qquad \beta_n^{kj} = b_{kj} - \frac{1}{n},$$

$$A_n^1(k, j) = \{x: \alpha_n^{kj} \leqq x < \alpha_{n-1}^{kj}\},$$

$$A_n^2(k, j) = \{x: \beta_{n-1}^{kj} \leqq x < \beta_n^{kj}\}.$$

The sets $A_n^i(k, j)$, $i=1, 2$; $n\geqq N_{kj}+1$ are disjoint and

Presented to the Society, April 17, 1959; received by the editors May 4, 1959 and, in revised form, July 2, 1958.

[1] This paper forms a part of the author's doctoral dissertation written at Iowa State College under the direction of Professor H. P. Thielman.

$$I_{kj} = \bigcup_{n=1}^{\infty} [\overset{1}{A}_n(k, j)\ \overset{2}{A}_n(k, j)].$$

For each of the sets $\overset{i}{A}_n(k, j)$ let $\overset{i}{A}\overset{kj}{E_n}$ be any measurable set contained in $\overset{i}{A}_n(k, j)$ for which $m(\overset{i}{A}\overset{kj}{E_n}) = \gamma m(\overset{i}{A}_n(k, j))$, $i = 1, 2$. Let $\overset{i}{A}\overset{kj}{F_n}$ be defined by $\overset{i}{A}\overset{kj}{F_n} = \overset{i}{A}_n(k, j) - \overset{i}{A}\overset{kj}{E_n}$. Then

$$m(\overset{i}{A}\overset{kj}{F_n}) = (1 - \gamma)m(\overset{i}{A}_n(k, j)).$$

Finally let

$$E_k = \bigcup_{j=1}^{\infty} \bigcup_{n>N_{kj}} (\overset{1}{A}\overset{kj}{E_n} \cup \overset{2}{A}\overset{kj}{E_n}),$$

$$F_k = \bigcup_{j=1}^{\infty} \bigcup_{n>N_{kj}} (\overset{1}{A}\overset{kj}{F_n} \cup \overset{2}{A}\overset{kj}{F_n}).$$

The sets E_k and F_k are disjoint and $T_k = E_k \cup F_k$.

Returning to the sets G_k restrict G_{k+1} so that

$$m(G_{k+1} \cap \overset{i}{A}\overset{kj}{E_n}) \leqq \frac{1}{n^k} m(\overset{i}{A}\overset{kj}{E_n}),$$

$$(1) \qquad m(G_{k+1} \cap \overset{i}{A}\overset{kj}{F_n}) \leqq \frac{1}{n^k} m(\overset{i}{A}\overset{kj}{F_n})$$

for $i = 1, 2$; $n \geqq N_{kj} + 1$; $j = 1, 2, \cdots$, which is possible since Z has measure zero.

The set S defined by

$$S = \bigcup_{k=1}^{\infty} (E_k - G_{k+1})$$

has density γ at every point of Z.

For, let z be any element in Z, and let h be the smallest positive integer such that $z \in Z_h \subset G_h$. Let I be an open interval containing z and contained in G_h. Restricting $m(I) < 1/2$ there exists an integer $p > 1$ such that

$$(2) \qquad \frac{1}{p+1} \leqq m(I) < \frac{1}{p}.$$

Since $z \in Z_h$, $z \notin T_h$ and by (2) if an end point of I falls in $\overset{i}{A}_n(h, j)$ then $n \geqq p$ and

$$m(\overset{i}{A}_n(h, j)) = \frac{1}{n(n-1)} \leqq \frac{2}{p(p+1)}.$$

The interval I consists of the following:

1. A set H composed of disjoint sets $A_n^i(h, j)$ where $i=1, 2$ and $n \geqq p$.

2. A set J which consists of two open or half open intervals, possibly empty, at the ends of I each of whose lengths does not exceed $2/p(p+1)$ so that $m(J) \leqq 4/p(p+1)$.

3. A set $N = Z \cap I$ of measure zero.

We will show first that

$$(3) \qquad \begin{aligned} (\gamma - 4/p)m(I) &\leqq m(H \cap E_h) \leqq \gamma m(I), \\ (1 - \gamma - 4/p)m(I) &\leqq m(H \cap F_h) \leqq (1 - \gamma)m(I). \end{aligned}$$

We have

$$m(H \cap E_h) = \sum m(A^i E_n^{hj}) = \gamma \sum m(A_n^i(h, j)) = \gamma m(H) \leqq \gamma m(I),$$

where the summation is taken over all n, j and i for which $A_n^i(h, j) \subset I$. Also, $m(H \cap F_h) = (1-\gamma)m(H) \leqq (1-\gamma)m(I)$.

From the maximum measure of J and inequality (2) we have $m(J) \leqq (4/p)m(I)$. Thus since $m(I) = m(H) + m(J)$, $m(H) \geqq (1-4/p)m(I)$. Therefore,

$$m(H \cap E_h) \geqq (\gamma - 4/p)m(I)$$

and

$$m(H \cap F_h) \geqq (1 - \gamma - 4/p)m(I).$$

Thus inequalities (3) are satisfied.

Next it will be shown that for all positive integers q

$$(4) \qquad \begin{aligned} m(G_{h+q} \cap H \cap E_h) &\leqq m(H \cap E_h)/p^q, \\ m(G_{h+q} \cap H \cap F_h) &\leqq m(H \cap F_h)/p^q. \end{aligned}$$

From the inequalities (1) and the fact that $n \geqq p$,

$$\begin{aligned} m(G_{h+q} \cap H \cap E_h) &= \sum m(G_{h+q} \cap A^i E_n^{hj}) \\ &\leqq \sum m(A^i E_n^{hj})/n^q \leqq m(H \cap E_h)/p^q \end{aligned}$$

where the summations are taken over all n, j, i for which $A_n^i(h, j) \subset I$. The second inequality in (4) is obtained in the same way. Since $G_{h+q} \supset E_{h+q}$, from (4) and (3) we obtain

$$m(E_{h+q} \cap H \cap E_h) \leqq \gamma m(I)/p^q,$$

$$m(E_{h+q} \cap H \cap F_h) \leqq (1 - \gamma)m(I)/p^q,$$

for $q=1, 2, \cdots$. Now

$$I = (H \cap E_h) \cup (H \cap F_h) \cup N \cup J,$$

so

$$\begin{aligned} m\left(I \cap \bigcup_{k=h}^{\infty} E_k\right) &\leqq m(H \cap E_h) + \sum_{k=h+1}^{\infty} m(E_k \cap H \cap E_h) \\ &\quad + \sum_{k=h+1}^{\infty} m(E_k \cap H \cap F_h) + m(J) \qquad (5) \\ &< \left(\gamma + \frac{5}{p-1}\right) m(I). \end{aligned}$$

Since $I \supset (H \cap E_h) \cup (H \cap F_h)$ and $E_h \cap F_h = \emptyset$,

$$I \cap (E_h - G_{h+1}) \supset (H \cap E_h) - (G_{h+1} \cap H \cap E_h)$$

and

$$m(I \cap (E_h - G_{h+1})) > (\gamma - 5/p) m(I). \qquad (6)$$

However,

$$I \cap (E_h - G_{h+1}) \subset I \cap S \subset I \cap \bigcup_{k=h}^{\infty} E_k$$

so that application of inequalities (5) and (6) gives

$$\gamma - \frac{5}{p} < \frac{m(I \cap S)}{m(I)} < \gamma + \frac{5}{p-1} \cdot$$

Bibliography

1. Goffman, Casper, *On Lebesgue's density theorem*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 384–387.

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