S^n, and K is then a fibre bundle over S^n. Consequently we have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccccc} \cdots \to & \Pi_k(O_n) & \to & \Pi_k(O_{n+1}) & \to & \Pi_k(S^n) & \to & \Pi_{k-1}(O_n) & \to & \Pi_{k-1}(O_{n+1}) & \to \cdots \\ & \downarrow i_* & & \downarrow i_* & & \downarrow j_* & & \downarrow i_* & & \downarrow i_* & \\ \cdots \to & \Pi_k(K_0) & \to & \Pi_k(K) & \to & \Pi_k(S^n) & \to & \Pi_{k-1}(K_0) & \to & \Pi_{k-1}(K) & \to \cdots \end{array}$$

where j_* is the identity. It is easily seen that $\mathrm{Ker}(i_*: \Pi_k(O_{n+1}) \to \Pi_k(K))$ is the image under $\Pi_k(O_n) \to \Pi_k(O_{n+1})$ of $\mathrm{Ker}(i_*: \Pi_k(O_n) \to \Pi_k(K_0))$. Since this last kernel is O the theorem follows.

References

1. D. Montgomery and L. Zippin, *Topological transformation groups*, New York, Interscience Publishers, 1955.

2. N. Steenrod, *The topology of fibre bundles*, Princeton, Princeton University Press, 1951.

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A NOTE ON GAUSS' FIRST PROOF OF THE QUADRATIC RECIPROCITY THEOREM

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We assume that the reader is familiar with Mathews' exposition [1, pp. 45–50] of the inductive proof of the reciprocity theorem. There are three main cases:

I. pRq,

II. pNq, $q \equiv 3 \pmod 4$,

III. pNq, $q \equiv 1 \pmod 4$.

In I we have $e^2 - p = qf$, in II we have $e^2 + p = qf$. In III we have first the lemma which asserts the existence of a prime $p' < q$ such that qNp'. This implies $p'Nq$, so that $pp'Rq$ and so $e^2 - pp' = qf$. In each of the cases I and II it is necessary to treat two sub-cases; in case III there are four sub-cases. Thus in all there are eight cases to consider.

We should like to point out in this note that it is possible to handle all cases simultaneously by introducing a little notation. To begin with, we define

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$$r = \begin{cases} 1 & \text{(case I)} \\ -1 & \text{(case II)} \\ p' & \text{(case III).} \end{cases}$$

Thus we have the single equation

$$e^2 - rp = qf, \tag{1}$$

where e is even and $<q, f$ is odd and $|f| < q$.

Next we put

$$d = (f, rp), \qquad f = df', \qquad e = de', \qquad rp = dd',$$

so that (1) becomes

$$de'^2 - d' = qf'; \tag{2}$$

moreover

$$(f', dd') = (d, d') = (q, dd') = 1.$$

From (2) we get $qf' \equiv -d' \pmod 4$, so that

$$q + d' + f' \equiv 1 \pmod 4. \tag{3}$$

Now from (2) we also get

$$\left(\frac{dd'}{f'}\right) = \left(\frac{qdf'}{d'}\right) = \left(\frac{-qd'f'}{d}\right) = 1,$$

which gives

$$\left(\frac{q}{dd'}\right) = \left(\frac{-d'}{d}\right)\left(\frac{d}{d'}\right)\left(\frac{f'}{dd'}\right).$$

We now apply the generalized reciprocity theorem:

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = -1^{(m-1)(n-1)/4},$$

where m and n are odd and relatively prime; also one of the numbers may be negative. The special cases m or $n = \pm 1$ are included. Then we get, since $(dd'/f') = 1$,

$$\left(\frac{q}{dd'}\right) = (-1)^{\lambda}, \tag{4}$$

where

$$\lambda = \frac{1}{4}(d-1)(-d'-1) + \frac{1}{4}(f'-1)(dd'-1).$$

Using (3) we find that

$$\begin{aligned}
\lambda &\equiv \frac{1}{4}(d-1)(-d'-1) - \frac{1}{4}(q+d')(dd'-1) \\
&\equiv \frac{1}{4}(d-1)(-d'-1) - \frac{1}{4}(d'+1)(d+d'-2) \\
&\quad - \frac{1}{4}(q-1)(dd'-1) \\
&\equiv -\frac{1}{4}(d'+1)(2d+d'-3) - \frac{1}{4}(q-1)(dd'-1) \\
&\equiv -\frac{1}{4}(d'^2-1) - \frac{1}{2}(d'+1)(d-1) - \frac{1}{4}(q-1)(dd'-1) \\
&\equiv \frac{1}{4}(q-1)(rp-1) \pmod 2;
\end{aligned}$$

where at the last step we used $rp = dd'$. Thus (4) becomes

$$\left(\frac{q}{rp}\right) = (-1)^{(q-1)(rp-1)/4}. \tag{5}$$

In cases I and II (5) is in obvious agreement with the reciprocity theorem; in III there is also agreement since we have qNp'. Thus in III (5) reduces to

$$\left(\frac{q}{p}\right) = -1,$$

which is the desired relation.

Reference

1. G. B. Mathews, *Theory of numbers*, Cambridge, 1892.

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