THE RELATION BETWEEN TWO DEFINITE INTEGRALS

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The C_rP-integral of Burkill [1] and the $\mathcal{P}^{r+1}$-integral of James [2] both depend on the concept of major and minor functions which in turn are defined in terms of generalized derivatives and derivates. The upper and lower C_r-derivates occurring in the definition of the C_rP-integral of $f(t)$, are given in terms of a $C_{r-1}P$-integral. The C_rP-integral is thus defined inductively, the C_0P-integral being the well known Perron integral [4, p. 201]. The definition of the $\mathcal{P}^{r+1}$-integral, however, is made directly, the generalized derivative utilized in this case being the de la Vallée Poussin derivative [6, p. 59]. The properties of the de la Vallée Poussin derivatives in connection with the C_rP-integral have been studied by Burkill [1] and Sargent [5], and James [2] was able to show that C_rP-integrability implies $\mathcal{P}^{r+1}$-integrability. In particular James proved that the indefinite $\mathcal{P}^{r+1}$-integral is equal to an $(r+1)$-fold integral in which the innermost integral is an indefinite C_rP-integral, the next one an indefinite $C_{r-1}P$-integral, and so on, the outermost integral being an indefinite C_0P-integral. This note demonstrates the factor, dependent on r, by which the C_rP-(definite) integral differs from the $\mathcal{P}^{r+1}$-(definite) integral.

THEOREM. *Let $f(t)$ be C_rP-integrable over $[-b, b]$, where r is a positive integer; define f for all other real t by extending it to be periodic with period $2b$. Let m be $(r-1)/2$ if r is odd and $(r-2)/2$ if r is even, and let $\alpha_1, \cdots, \alpha_{r+1}$ be the points*

$$-2(m+1)b, -2mb, \cdots, -2b, 2b, 4b, \cdots, 2(r-m)b.$$

Then

$$(2b)^{-r}\binom{r+1}{m+1}\int_{\alpha_i}^{0} f(t)d_{r+1}t = C_rP\int_{-b}^{b} f(t)dt.$$

PROOF. Let

$$F_r(x) = C_rP\int_{-rb}^{x} f(t)dt,$$

$$F_k(x) = C_kP\int_{-rb}^{x} F_{k+1}(t)dt, \qquad 0 \leqq k \leqq r-1,$$

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$$F(x) = F_0(x),$$

and consider the case when r is odd. (The proof for r even is similar.) It is easy to verify that

$$F_r(x) = C_r P \int_{-rb}^{x} f(t)dt = \frac{x}{2b} C_r P \int_{-b}^{b} f(t)dt + G(x) = x\mu + G(x),$$

where $G(x)$ is periodic with period $2b$. It follows that $F(x)$ may be written as $G_1(x)+G_2(x)+G_3(x)$ where $G_1(x)=\mu x^{r+1}/(r+1)!$, $G_2(x)$ is periodic with period $2b$, and $G_3(x)$ is a polynomial in x of degree r.

It is known [2, p. 168] that if $\lambda(x;\alpha_i)=\prod_{j\neq i}(x-\alpha_j)/(\alpha_i-\alpha_j)$, then

$$(*) \qquad (-1)^{[(r+1)/2]+r+1} \int_{(\alpha_i)}^{0} f(t)d_{r+1}t = F(0) - \sum_{i=1}^{r+1} \lambda(0;\alpha_i)F(\alpha_i).$$

The terms involving G_1 in $(*)$ are equal to $\prod_{i=1}^{r+1}(-\alpha_i)$ multiplied by a divided difference [3, Chapter 1] of order $(r+1)$ for the function G_1. Every divided difference of order $(r+1)$ for the constant $G_2(0)$ and the polynomial $G_3(x)$ equals 0.

The right hand side of $(*)$ thus reduces to

$$(-1)^{(r+1)/2}\mu(2b)^{r+1} \Big/ \binom{r+1}{m+1}.$$

This yields

$$(2b)^{-r-1}\binom{r+1}{m+1}\int_{(\alpha_i)}^{0} f(t)d_{r+1}t = (2b)^{-1}C_r P\int_{-b}^{b} f(t)dt,$$

the required result.

References

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