RECURSION OF SET TRAJECTORIES IN A TRANSFORMATION GROUP[1]

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1. **Introduction.** In [3][3] and [4] Trjitzinsky considers a one parameter group of homeomorphisms acting on a metric space. He considers an equivariant mapping which assigns to each point of the space a closed set which contains that point, and then introduces the notion of recurrence (stability). Here we consider a transformation group (X, T, π), where X is a metric space with the metric d; T is a topological group; and π is a continuous mapping of $T\times X$ onto itself satisfying the following: If e is the identity of T then $\pi(e, x)=ex=x$, and if t and $t'\in T$ and $x\in X$ then $\pi(t, \pi(t', x))=t(t'x)=(tt')x=\pi(tt', x)$. The group T is allowed to act on the subsets of X and we discuss continuity, recursion, and a conjecture raised in [4, p. 99].

NOTATION. $A(X)=\{E\subset X: E\neq\emptyset\}$, $B(X)=\{E\subset X: E\neq\emptyset,\ \bar{E}$ compact$\}$, $K(X)=\{E\subset X: E\neq\emptyset$ and closed$\}$, $C(X)=\{E\subset X: E\neq\emptyset$ and compact$\}$.

DEFINITION 1.1. If (X, d) is a metric space then the Hausdorff pseudo-metric h on $A(X)$ induced by d is given by

$$h(D, E) = \max(h_1(D, E), h_1(E, D))$$

where

$$h_1(D, E) = \sup\{d(x, E): x \in D\}.$$

If h is restricted to $K(X)$ or $C(X)$ then h is a metric. The remainder of the discussion will concern itself primarily with $K(X)$ and $C(X)$. The results obtained may be applied to $A(X)$ and $B(X)$ after making the observation that $K(X)$ and $C(X)$ are homeomorphic to the quotient spaces obtained from $A(X)$ and $B(X)$ when one identifies a set with its closure.

DEFINITION 1.2. If (X, T, π) is a transformation group then in the triple $(A(X), T, \pi)$ $[(B(X), T, \pi), (K(X), T, \pi), (C(X), T, \pi)]$ we let $\pi(t, E)=t(E)=\{tx: x\in E\}$ for $E\in A(X)$ $[B(X), K(X), C(X)]$.

2. **Continuity.** In the following sections we will always have in mind a transformation group (X, T, π). If $t\in T$ then the t transition of X $[K(X), C(X)]$ will be the mapping

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$$\pi_t: X \to X\ [K(X) \to K(X),\ C(X) \to C(X)]$$

defined by $\pi_t(x) = tx$ for $x \in X$ $[\pi_t(E) = t(E)$ for $E \in C(X)$ or $K(X)]$.

THEOREM 2.1. *If each t transition of X is uniformly continuous then so is each t transition of $K(X)$.*

PROOF. [**2**, p. 170].

THEOREM 2.2. *If X and T are locally compact then $(C(X), T, \pi)$ is a transformation group.*

PROOF. The homomorphism and identity properties are clear and we need only show that π is continuous on $T \times C(X)$. Let $(t_0, E) \in T \times C(X)$ and let $\epsilon > 0$ be arbitrary. Since X is locally compact there is an $\eta > 0$ such that $E(\eta) = \{x: d(x, E) \leqq \eta\}$ is compact and in T there is a compact neighborhood V of e. It follows that $t_0 V \times E(\eta)$ is a compact neighborhood of (t_0, E) in $T \times X$ and thus π is uniformly continuous on this neighborhood. Consequently there is a $\delta > 0$, and $< \eta$, and a neighborhood U of e, $U \subset V$, such that if $t \in t_0 U$, and x and $y \in E(\eta)$, such that $d(x, y) < \delta$, then $d(tx, t_0 y) < \epsilon/2$. If $D \in C(X)$ is chosen so that $h(E, D) < \delta$ and $t \in t_0 U$ then $h_1(tD, t_0 E) = \sup\{d(tx, t_0 E): tx \in tD\} < \epsilon$ and similarly $h_1(t_0 E, tD) < \epsilon$.

COROLLARY 2.3. *If π is uniformly continuous on $T \times X$ then $(K(X), T, \pi)$ is a transformation group.*

THEOREM 2.4. *If (X, d') is a separable, locally compact, metric space and if T is locally compact, then there is a metric d equivalent to d' such that $(K(X), T, \pi)$ is a transformation group if $K(X)$ has the Hausdorff metric h induced by d.*

PROOF. Since X is locally compact and separable we can express X as the countable union of compact sets. It follows that if $\hat{X}$ is the one point compactification of X then $\hat{X}$ has a countable base and is thus metrizable by a metric d. The injection $i: X \to \hat{X}$ is a homeomorphism into $\hat{X}$ and thus d is equivalent to d' and X may be metrized by d. Let h be the Hausdorff metric induced by d on $K(X)$ and $K(\hat{X})$. Letting $t(\infty) = \infty$ for all $t \in T$ and applying Theorem 2.2 we see that $(K(\hat{X}), T, \pi)$ is a transformation group. We define the mapping $j: K(X) \to K(\hat{X})$ by $j(E) = \mathrm{Cl}\{i(E)\} = \mathrm{Cl}\{i(x): x \in E\}$ which indicates closure in $\hat{X}$. We see that j is an isometry since

$$h(D, E) = h(i(D), i(E)) = h(\mathrm{Cl}\{i(D)\}, \mathrm{Cl}\{i(E)\}) = h(j(D), j(E))$$

and thus $(K(X), T, \pi)$ is a transformation group since $tj = jt$ for all $t \in T$ and $h(tE, tD) = h(tj(E), tj(D))$.

COROLLARY 2.5. *If (X, d') is a locally compact, separable, metric space then there is a metric d equivalent to d' such that each t transition of $(K(X), h)$ is continuous where h is the Hausdorff metric induced by d.*

3. **Recursion.** Following Gottschalk and Hedlund [1, p. 21] we let $\mathfrak{A}$ be a distinguished class of subsets of T which are called admissible.

DEFINITION 3.1. T is recursive at $E\in A(X)$ if for each $x\in E$ and each neighborhood U of x, $U\subset X$, there is an admissible set $S\subset T$ such that $sE\cap U\neq\emptyset$ for each $s\in S$.

DEFINITION 3.2. T is regionally recursive at $E\in A(X)$ if for each $x\in E$ and each neighborhood U of x and V of E, U and $V\subset X$, there is an admissible set $S\subset T$ such that $sV\cap U\neq\emptyset$ for each $s\in S$.

We will say that T is recursive [regionally recursive] on $A(X)$, $B(X)$, $C(X)$ or $K(X)$ if T is recursive [regionally recursive] at each point of $A(X)$ or $B(X)$, $C(X)$ or $K(X)$ respectively. If in the above definitions E reduces to a single point $\{x\}$ then the definitions are those given in [1, p. 21]. It is clear that T is recursive [regionally recursive] on X if and only if T is recursive [regionally recursive] on $A(X)$.

For the remainder of the paper we let $\{P(i)\}$, $i\in I=\{1, 2, 3 \cdots\}$, be a sequence of subsets of T and define an admissible set to be one which meets each $P(i)$.

THEOREM 3.1. *If each t transition of $C(X)$ is continuous and if R is the set of T recursive points of $C(X)$ then,*

(a) *R is a G_δ subset of $C(X)$.*

(b) *If T is regionally recursive on X, R is a residual subset of $C(X)$.*

PROOF. If we let $E(n, m)$ be the collection of all $E\in C(X)$ for which there is an $x\in E$ such that $d(x, tE)\geqq 1/m$ for all $t\in P(n)$, then $R=C(X)-\bigcup_{n,m=1} E(n, m)$. If $\{E_j\}$, $j\in I$, is a sequence in $E(n, m)$ such that $E_j\to E$ then in each E_j we let x_j be chosen so that $d(x_j, tE_j)\geqq 1/m$ for all $t\in P(n)$. For each $j\in I$ there is a $y_j\in E$ such that $d(x_j, y_j)\to 0$ and since E is compact there is an infinite subset $I'\subset I$ and a $y\in E$ such that if k ranges over I' then $y_k\to y$ and consequently $x_k\to y$. If $E\notin E(n, m)$ then there is a $t\in P(n)$ for which $d(y, tE)=1/m-\eta$ where $\eta>0$. Since t is continuous on $C(X)$ there is a $\delta>0$ such that if $h(E, D)<\delta$ then $h(tE, tD)<\eta/4$ and there is a $k\in I'$ such that $d(y, x_k)<\eta/4$. It follows that

$$\begin{aligned} d(x_k, tE_k) &\leqq d(x_k, y) + d(y, tE_k) \\ &\leqq d(x_k, y) + d(y, tE) + \frac{\eta}{2} \leqq \frac{1}{m} - \frac{\eta}{4} < \frac{1}{m}. \end{aligned}$$

This is clearly impossible and it follows that each $E(n, m)$ is closed and (a) is proved.

Assume now that T is regionally recursive on X. If $H(n, m)$ is the set of all $x \in X$ for which there is a $t \in P(n)$ such that $d(x, tx) < 1/m$ then each $H(n, m)$ is open and dense in X [1, p. 25]. If $\epsilon > 0$ is arbitrary and if $E \in E(n, m)$ we cover E with a finite number of spheres of radius ϵ. In each such sphere there is an $x_j \in H(n, m)$. If $D = \bigcup x_j$ then D is compact and $h(E, D) < \epsilon$ and $D \notin E(n, m)$. It follows that each $E(n, m)$ is nowhere dense and (b) is proved.

Lemma 3.2. *If $U \subset X$ is a dense open set such that $X - U \neq \emptyset$ and if $\mathfrak{E} = \{E: E \in K(X)$ and $d(E, X - U) > 0\}$ then $\mathfrak{E}$ is open and dense in $K(X)$.*

Proof. Clearly $\mathfrak{E}$ is open. If $\epsilon > 0$ is arbitrary and if $D \in K(X)$, then if we let $E = \{x: d(x, D) < \epsilon\}$ it suffices to show that $\overline{E} \in \overline{\mathfrak{E}}$. Let $F = E \cap U$ and let $U(n) = \{x: d(x, X - F) \geqq 1/n\}$. Since U is dense and open we have $h(\overline{E}, F) = 0$ and $F = \bigcup_{n \in I} U(n)$ and $U(n) \in \mathfrak{E}$. It follows that $h(\overline{E}, U(n)) = h(F, U(n)) \to 0$ as $n \to \infty$.

Theorem 3.3. *If each t transition of $K(X)$ is continuous and if T is regionally recursive on X and if R is the set of recursive points of $K(X)$ then R is a residual subset of $K(X)$.*

Proof. If $H(n, m)$ is defined as in Theorem 3.1 then each $H(n, m)$ is a dense open subset of X and by Lemma 3.2 each $K(H(n, m))$ contains a dense open subset of $K(X)$. If $E \in \bigcap_{m,n=1} K(H(n, m))$ then $E \in R$ since if $x \in E$ and U is a neighborhood of x there is an m such that $S(x, 1/m) \subset U$ and thus for each $P(n)$ there is a t_n such that $U \cap t_n E \neq \emptyset$. The set $\{t_n\}$, $n \in I$, is extensive. It follows that $R \supset \bigcap_{m,n=1} K(H(n, m))$ and so R is a residual.

It is important to note that if X is complete then so is $C(X)$ and $K(X)$ or if X is locally compact then so is $C(X)$ [2, p. 161]. In these cases the residuals of $C(X)$ and $K(X)$ are dense.

4. **Recursion and an equivariant mapping.** In this section ϕ will be a continuous mapping of X into $K(X)$ such that $x \in \phi(x)$ for all $x \in X$, and $t\phi = \phi t$ for all $t \in T$.

Definition 4.1. ϕT is regionally recursive on X if for each $x \in X$ and each neighborhood U of x there is an admissible set $S \subset T$ such that

$$\bigcup_{y \in U} \phi(y) \cap s \bigcup_{y \in U} \phi(y) \neq \emptyset \qquad \text{for all } s \in S.$$

Definition 4.2. ϕT is almost recursive at x if for each neighborhood $U \supset \phi(x)$ there is an extensive set $S \subset T$ such that $U \cap s\phi(x) \neq \varnothing$ for all $s \in S$.

We remark that since $d(E, D) \leqq h(E, D)$ for $E, D \in A(X)$ it follows that $h(\phi(x), \phi(y))$ is simultaneously continuous in x and y.

THEOREM 4.3. *If $R \subset X$ is the set of almost recursive points then*
(a) *R is a G_δ subset of X.*
(b) *If ϕT is regionally recursive then R is a residual subset of X.*

PROOF. If $E(n, m)$ is the set of all $x \in X$ such that $d(\phi(x), t\phi(x)) \geqq 1/m$ for all $t \in P(n)$ then $E(n, m)$ is closed since if a sequence $\{x_j\}$ of $E(n, m)$ converges to x then $d(\phi(x_j), t\phi(x_j))$ converges to $d(\phi(x), t\phi(x))$ for $t \in T$. If there is an $E(n, m)$ which is somewhere dense then there is an $\epsilon > 0$, $0 < \epsilon < 1/m$, and a $y \in E(n, m)$ such that the ϵ sphere about y, $S(y, \epsilon) \subset E(n, m)$. Choose $\delta > 0$ and $< \epsilon$ such that if $x \in S(y, \delta)$ then $h(\phi(x), \phi(y)) < \epsilon/2$. Since ϕT is regionally recursive there is a $t \in P(n)$ such that $\bigcup_{z \in S(y,\delta)} \phi(z) \cap t \bigcup_{z \in S(y,\delta)} \phi(z) \neq \varnothing$. It follows that there are z and $w \in S(y, \delta)$ for which $\phi(z) \cap t\phi(w) \neq \varnothing$ and $h(\phi(z), \phi(w)) \leqq h(\phi(z), \phi(y)) + h(\phi(y), \phi(w)) < \epsilon$. If $u \in \phi(z) \cap t\phi(w)$ then

$$\begin{aligned} d(\phi(w), t\phi(w)) \leqq d(u, \phi(w)) &\leqq \sup\{d(v, \phi(w)) : v \in \phi(z)\} \\ &\leqq h(\phi(z), \phi(w)) < \epsilon < \frac{1}{m} \end{aligned}$$

and therefore $w \notin E(n, m)$ but this contradicts $w \in S(y, \delta) \subset E(n, m)$ and thus $E(n, m)$ is nowhere dense. Since $R = X - \bigcup_{n,m=1} E(n, m)$ our conclusions follow.

In [4, p. 99] Trjitzinsky considered T to be a one parameter group and defined $L^+(\phi(x))$ to be the set of all points $y \in X$ such that $\liminf_{t \to +\infty} d(y, t\phi(y)) = 0$. He conjectured that if for all x in a dense subset of X we have $L^+(\phi(x)) \supset \phi(x)$ then there is a residual such that for all x in this set $\liminf_{t \to +\infty} d(x, t\phi(x)) = \liminf_{t \to -\infty} d(x, t\phi(x)) = 0$. If we let $P(i) = (i, +\infty)$ and $P(-i) = (-\infty, -i)$ for $i \in I$ then Theorem 4.2 proves this conjecture.

If the fact that $x \in \phi(x)$ is dropped then Trjitzinsky's conjecture is false because ϕT may not be regionally recursive. Consider the following example. Let $Z = C \times C$ where C is the unit circle. Let (Z, T, Γ) be the transformation group given by the solution to the differential equations

$$\frac{dx}{dt} = a(x^2 + y^2), \qquad \frac{dy}{dt} = b(x^2 + y^2),$$

where x and y are coordinates modulo one and a/b is irrational. There is a point $z \in Z$ such that Tz is Poisson stable $(+)$ but not $(-)$. The set of points which are Poisson stable $(+)$ and $(-)$ are dense in Z.

In E_3 with coordinates (ξ, η, τ) consider the cylinder

$$Y = \{y : y = (\xi, \eta, \tau), \xi^2 + \eta^2 = 1\}.$$

Define the transformation group (Y, T, Λ) as follows:

$$\Lambda(t, y) = t(y) = t(\xi, \eta, \tau) = (\xi, \eta, \tau + t).$$

Imbed Z in E_3 such that $Z \cap Y = \emptyset$ and let $X = Z \cup Y$. We now define (X, T, π) and $\phi: X \to C(X)$ as follows:

$$\begin{aligned} \pi(t, x) &= \Gamma(t, x) && \text{for } x \in Z, \\ \pi(t, x) &= \Lambda(t, x) && \text{for } x \in Y, \\ \phi(x) &= x && \text{for } x \in Z, \\ \phi(x) &= \pi(\tau, z) = \tau(z) && \text{for } x \in Y \text{ and } x = (\xi, \eta, \tau). \end{aligned}$$

All the conditions of Trjitzinsky's conjecture are satisfied except $x \in \phi(x)$ and for each $x \in Y$ we have $\liminf_{t \to -\infty} d(t\phi(x), \phi(x)) \neq 0$ and Y is category II in X.

The hope that the conclusion of Theorem 4.2 can be strengthened to read "then ϕT is recursive on a residual" is in vain. Let $X = E_2$ with coordinates $x = (x_1, x_2)$ and define (X, T, π) for T the additive group of the reals as follows: $\pi(t, x) = (x_1 e^t, x_2 e^t)$. If we define $\phi(x) = \{y : y_1^2 + y_2^2 \leq x_1^2 + x_2^2\}$ then ϕT is regionally recursive on X, where an admissible set is one that meets each open infinite interval, but for $x \neq (0, 0)$ we have $L^-(\phi(x)) \subset \phi(x)$ properly.

Bibliography

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