UNIVERSAL MINIMAL SETS[1]

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Let (X, T) and (Y, T) be transformation groups with the same phase group T. A *homomorphism* $f: X \to Y$ is a continuous mapping such that $(xt)f = (xf)t$ $(x \in X,\ t \in T)$. A transformation group (X, T) is called a *minimal set* if $(xT)^- = cls[xt/t \in T] = X(x \in X)$. In this note it will be proved that given any abstract group T there exists a minimal set (M, T) with compact phase space M such that any minimal set (X, T) with compact X is a homomorphic image of (M, T). Furthermore this "universal minimal set" is unique up to an isomorphism, and given $x \in M$, $t \in T$ with $t \neq e$ then $xt \neq x$. For a more complete discussion of several notions involved above see [**2**] and [**3**].

DEFINITION 1. *The β-compactification as a transformation group.* Let T be a discrete group, let βT be the β-compactification of T, and let $t \in T$. Then the map $s \to st$ of T into βT is continuous and so may be extended to a map of βT into βT. Thus each element of T may be identified with a homeomorphism of βT onto βT. Under this identification $(\beta T, T)$ becomes a transformation group.

Henceforth all transformation groups (X, T) will be assumed to have compact phase spaces, X, and discrete phase group T.

LEMMA 1. *Let (X, T) be a transformation group, let $x \in X$. Then there exists a homomorphism f mapping $(\beta T, T)$ onto $((xT)^-, T)$.*

PROOF. The mapping $t \to xt$ $(t \in T)$ of T into X is uniformly continuous, since xT is totally bounded. Hence there exists a continuous function f mapping βT onto $(xT)^-$ with $tf = xt$ $(t \in T)$. Hence $(ts)f = (tf)s$ $(t, s \in T)$ and so by continuity $(ys)f = (yf)s$ $(y \in \beta T,\ s \in T)$. The proof is completed.

COROLLARY 1. *Let (X, T) be minimal, and let M be a minimal subset of βT. Then (X, T) is a homomorphic image of (M, T).*

DEFINITION 2. *Universal minimal set associated with a group T.* A transformation group (M, T) will be called a *universal minimal set associated with T* if M is minimal and if any minimal set (X, T) is a homomorphic image of (M, T).

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Corollary 1 shows that any minimal subset of $(\beta T, T)$ is a universal minimal set associated with T.

In order to prove that all universal minimal sets associated with a given group T are isomorphic I must make use of the following theorem which is proved in [1, Lemma 5].

THEOREM 1. *Let (X, T) be a transformation group, let (I, T) be the transformation group associated with a minimal right ideal I of the enveloping semigroup $E(X, T)$, and let f be a homomorphism of I onto I. Then f is an isomorphism onto.*

LEMMA 2. *Let (M, T) be a universal minimal set, and let I be a minimal right ideal in $E(M, T)$. Then (M, T) and (I, T) are isomorphic.*

PROOF. Let $x \in M$. Then the map $\pi_x: I \to M$ such that $p\pi_x = xp\,(p \in I)$ is a homomorphism onto. Since (M, T) is universal and (I, T) is minimal, there exists a homomorphism f of (M, T) onto (I, T). Hence $\pi_x f$ is a homomorphism of (I, T) onto (I, T). By Theorem 1 this map is an isomorphism onto, whence π_x is one-one. The proof is completed.

COROLLARY 1. *Let (M, T) be a universal minimal set and let f be a homomorphism of M into M. Then f is an isomorphism onto.*

PROOF. The map f is onto since M is minimal. Now let I be a minimal right ideal in $E(M, T)$ and let $x \in M$. Then by Lemma 2, $\pi_x f \pi_x^{-1}$ is a well defined homomorphism of (I, T) onto (I, T). By Theorem 1, $\pi_x f \pi_x^{-1}$ is one-one, hence so is f.

THEOREM 2. *All universal minimal sets associated with the group T are isomorphic.*

PROOF. Let (M, T) and (N, T) be universal minimal sets. Then there exist homomorphisms f, g of M onto N and N onto M respectively. Hence fg is a homomorphism of M onto M which by Corollary 1 to Lemma 2 must be one-one. Hence f is one-one. The proof is completed.

DEFINITION 3. Let (X, T) be a transformation group. The action of T on X is said to be *strongly effective* if given $x \in X$ and $t \in T$ with $t \neq e$, then $xt \neq x$.

In order to show that the action of T on its associated universal set is strongly effective I must make use of the identification of βT with the set of ultrafilters on T, see [4]. Let $\mathfrak{U}$ be an ultrafilter on T and $t \in T$. Then the image of $\mathfrak{U}$ under t is the ultrafilter $\mathfrak{U}t = [Ut / U \in \mathfrak{U}]$.

THEOREM 3. *Let (M, T) be the universal minimal set associated with T. Then the action of T on M is strongly effective.*

PROOF. Let $t \in T$ with $t \neq e$ and let $\mathfrak{U}$ be an ultrafilter on T. I shall show that $\mathfrak{U}t \neq \mathfrak{U}$.

Let $\mathfrak{F} = [F/F \subset T$ and $Ft \cap F = \phi]$. Then $[e] \in \mathfrak{F}$ implies that $\mathfrak{F} \neq \phi$. Furthermore if $\mathfrak{F}$ is ordered by inclusion, it is inductive. Let F be a maximal element of $\mathfrak{F}$.

Now let $x \in T$ and suppose $x \in Ft \cup F$. Then $G = F \cup \{x\}$ is not in $\mathfrak{F}$ and so $Gt \cap G \neq \phi$; i.e. $(Ft \cup \{xt\}) \cap (F \cup \{x\}) \neq \phi$. Since $F \in \mathfrak{F}$, this means that $xt \in F$. Thus $T = F \cup Ft \cup Ft^{-1}$.

Since $\mathfrak{U}$ is an ultrafilter on T, one of the sets F, Ft, Ft^{-1} must be in $\mathfrak{U}$. If $\mathfrak{U}t$ were equal to $\mathfrak{U}$, then $\mathfrak{U}t^{-1}$ would also be equal to $\mathfrak{U}$. This would imply that as soon as one of the sets F, Ft, Ft^{-1} were in $\mathfrak{U}$ they would all be in $\mathfrak{U}$. This is impossible since $F \cap Ft = \phi$. The proof is completed.

Let $B(T)$ be the set of functions on T to the unit interval provided with the topology of pointwise convergence. Let $f \in B(T)$ and $t \in T$. Then one may define the element ft of $B(T)$ in two ways; (1) $s(ft) = (ts)f(s \in T)$ and (2) $s(ft) = (st^{-1})f(s \in T)$. In this way one obtains two transformation groups with phase group T. These will be denoted $B_1(T)$, $B_2(T)$.

THEOREM 4. *Let $t, s \in T$ with $t \neq s$. Then there exist functions $f, g \in B(T)$ such that*

1. $f(t) \neq f(s)$, $g(t) \neq g(s)$.
2. *f is an almost periodic point of $B_1(T)$ and g is an almost periodic point of $B_2(T)$.*
3. *The range of f = range of g = the two element set $\{0, 1\}$.*

PROOF. Let $(M \cdot T)$ be the universal minimal set associated with T and let $x \in M$. By Theorem 3 $xt \neq xs$. Since M is totally disconnected [4], there exists an open-closed subset U of M such that $xt \in U$ and $xs \notin U$. Let h be the characteristic function of U and set $rf = (xr)h$ $(r \in T)$. Then f clearly satisfies 1 and 3.

To show that f satisfies 2, let $\epsilon > 0$ and $s_1, \cdots, s_n$ be elements of T. Then

$$(f; s_1, \cdots, s_n; \epsilon) = [u/u \in B(T) \quad \text{and} \quad |s_iu - s_if| < \epsilon i = 1, \cdots, n]$$

is a typical neighborhood of f. Let α be an index of M such that $(a, b) \in \alpha$ implies $|ah - bh| < \epsilon$, and let β be an index on M such that $(a, b) \in \beta$ implies that $(as_i, bs_i) \in \alpha$ for $i = 1, \cdots, n$. Finally let $A = [r/r \in T$ and $(xr, x) \in \beta]$. Then A is a syndetic [3] subset of T, and

$r\in A$ implies that $|s_i(fr)-s_if| = |(rs_i)f-s_if| = |(xrs_i)h-(xs_i)h| <\epsilon$, $i=1, \cdots, n$ since $(xr, x)\in\beta$. Consequently $fr\in(f; s_i, \cdots, s_n; \epsilon)$ $(r\in A)$. This completes the proof for $B_1(T)$.

To obtain g, one replaces the minimal set M in the above argument by the universal minimal set associated with the group T^* opposite to T (i.e. T^* is T provided with the group operation "o" where $s \circ t = ts$ $(t, s\in T)$).

Theorem 4 states that the points of any group T may be separated by almost periodic points of $B(T)$. This is in marked contrast to the situation which prevails if one demands that the points of T be separated by almost periodic functions in the sense of von Neumann. There exist groups T [5] on which the only almost periodic functions are the constants.

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