CONTINUED FUNCTION EXPANSIONS OF REAL NUMBERS[1]

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1. **Introduction.** We present a theory of continued function expansions of numbers which contains the generalized continued fractions of B. H. Bissinger [1] and the generalized decimal representations of C. J. Everett [2]. The latter used the following algorithm for representing numbers as sequences of integers: for any $\gamma \geqq 0$ let $\gamma_0 = \gamma$, $\gamma_{n+1} = f^{-1}(\gamma_n - a_n)$, where $a_n = [\gamma_n]$ and f is strictly increasing and continuous from $[0, p]$ onto $[0, 1]$, p an integer. We generalize this, in particular, by admitting a wider class of functions than those of the form $f^{-1}(x-n)$. O. W. Rechard [3] gave a necessary and sufficient condition that the correspondence between numbers and sequences resulting from Everett's algorithm be 1-1. This condition appears in our theory as a simple functional relation similar to one considered by Schreier and Ulam [4].

2. **The algorithm.** The correspondence between numbers and sequences which we are going to describe depends on a collection of intervals and on functions defined on those intervals. More precisely

DEFINITION. *An algorithm frame, A,* consists of the following: an interval R; a subset P of the integers containing at least two integers; a partition of R into disjoint intervals I_n, $n \in P$; a subset P_0 of P containing at least two integers such that $I = \bigcup_{n \in P_0} I_n$ is an interval; intervals M_n, $n \in P$, homeomorphic to each other such that $M_n \subset I_n$ and $I_n - M_n$ consists of at most one point; and an interval M homeomorphic to each M_n such that $\bigcup_{n \in P_0} M_n \subset M \subset I$.

It follows from the above definition that if $\{M_n, n \in P\}$ is part of an algorithm frame then either all the M_n are open intervals or all are closed on one end, not necessarily the same, because not all the I_n can be closed and the M_n are homeomorphic to each other. Also, if any interval is infinite at some end it is taken to be open at that end.

DEFINITION. *An algorithm basis* consists of an algorithm frame A and a collection of homeomorphisms h_n, $n \in P$, mapping M_n onto M. We usually identify an algorithm basis by the couple (A, h_n).

Corresponding to any algorithm basis we have the following *algorithm* for relating points in R to sequences (finite or infinite) of integers:

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Let $x_0 \in R$.

$a(0)$: $\{a(0)$ is determined by the requirement that $x_0 \in I_{a(0)}.\}$

$\left\{\begin{array}{l}\text{If } x_0 \notin M_{a(0)}, \text{ stop, and represent } x_0 \text{ by the sequence of}\\ \text{one element } \{a(0)\}.\end{array}\right\}$

$a(1)$: $\left\{\begin{array}{l}\text{Since } x_0 \in M_{a(0)} \text{ we can let } x_1 = h_{a(0)}(x_0). \text{ Then } a(1)\\ \text{is determined by the requirement that } x_1 \in I_{a(1)}.\\ \text{Furthermore, } a(1) \in P_0 \text{ since } x_1 \in M \subset I.\end{array}\right\}$

$\left\{\begin{array}{l}\text{If } x_1 \notin M_{a(1)}, \text{ stop, and represent } x_0 \text{ by the sequence of}\\ \text{two elements } \{a(0), a(1)\}.\end{array}\right\}$

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$a(k)$: $\left\{\begin{array}{l}\text{Since } x_{k-1} \in M_{a(k-1)} \text{ we can let } x_k = h_{a(k-1)}(x_{k-1}).\\ \text{Then } a(k) \text{ is determined by the requirement that}\\ x_k \in I_{a(k)}, \text{ and } a(k) \in P_0.\end{array}\right\}$

$\{$If $x_k \notin M_{a(k)}$, stop, and represent x_0 by $\{a(0), \cdots, a(k)\}.\}$

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This algorithm contains the expansions considered by Bissinger and Everett. Let AB be the following algorithm basis:

$$R = [0, \infty),\ P = \{0, 1, 2, \cdots\},\ P_0 = \{1, 2, \cdots\},\ I_n = [n, n+1),$$
$$M_n = (n, n+1),\quad I = [1, \infty),\quad M = (1, \infty),$$

and let $h_n(x) = f^{-1}(x-n)$ for $x \in (n, n+1)$ where f is a continuous strictly decreasing function mapping $[1, \infty)$ onto $(0, 1]$. This contains Bissinger's expansions. Everett's expansions come from the algorithm basis AE given by:

$$R = I = M = [0, p),\qquad I_n = M_n = [n, n+1),$$
$$P = P_0 = \{0, 1, \cdots, p-1\},$$

and $h_n(x) = f^{-1}(x-n)$, $x \in [n, n+1)$ where f is continuous and strictly increasing from $[0, p]$ onto $[0, 1]$.

3. **1-1 Correspondence.** Given an algorithm basis (A, h_n), the algorithm defines a function h from R into the space C of finite or infinite sequences of integers $c = \{c(0), c(1), \cdots\}$ as follows: let x

yield c under the algorithm, then $h(x)=c$. Let E be the set of all such functions. In general we will use the convention that if $g\in E$ then the homeomorphisms in its algorithm basis are g_n.

DEFINITION. Let (A, h_n), (B, g_n) be algorithm bases. The corresponding functions h and $g\in E$ are said to be *equivalent*, written $h\sim g$, if A and B are identical and if h_n has the same sense as g_n for each n. (By this we mean that if h_n is monotonic increasing so is g_n and if h_n is monotonic decreasing so is g_n. This is *not* meant to imply that the sense of h_n is independent of n.)

Denote by $C(h)$ the range of h for $h\in E$.

The following theorems characterize the equivalent 1-1 functions in E:

THEOREM 1. *If $h\sim g$ and h is* 1-1 *onto $C(h)$ then $C(h)\subset C(g)$.*

COROLLARY 1. *If $h\sim g$, a finite sequence is in $C(h)$ if and only if it is in $C(g)$.*

COROLLARY 2. *If g is* 1-1, *$C(h)=C(g)$.*

NOTATION. A sequence of functions $hg\cdots k$ always means the composite function $h(g(\cdots(k)\cdots))$.

THEOREM 2. *Let g be* 1-1 *from R onto $C(g)$ and let h have the same algorithm frame as g. Then $h\sim g$ and h is* 1-1 *from R onto $C(g)$ if and only if there exists an increasing homeomorphism F from R onto R, which also maps M_n onto M_n for all n, such that $h_n^{-1}=F^{-1}g_n^{-1}F$.*

The following theorems are an application of Theorem 2 to bases AB and AE, respectively.

THEOREM 3. *Let (A, h_n) be an algorithm basis of the form AB. Let $h_n(x)=\hat{h}^{-1}(x-n)$. Then h is* 1-1 *if and only if there exists an increasing homeomorphism F mapping $[0, \infty)$ onto itself such that $F(x)=n+F(x-n)$ for $x\in[n, n+1)$ and $\hat{h}^{-1}(\tau)=F^{-1}(1/F(\tau))$ for all $\tau\in(0, 1]$.*

THEOREM 4. *Let (A, h_n) be an algorithm basis of the form AE. Let $h_n(x)=\hat{h}^{-1}(x-n)$. Then h is* 1-1 *if and only if there exists an increasing homeomorphism F mapping $[0, p]$ onto itself such that $F(x)=n+F(x-n)$ for $x\in[n, n+1)$ and $\hat{h}^{-1}(\tau)=F^{-1}(p\cdot F(\tau))$ for all $\tau\in[0, 1]$.*

Rechard's condition is that h is 1-1 if and only if there exists an increasing homeomorphism G mapping $[0, 1]$ onto itself such that $\hat{h}(y)=G^{-1}((n+G(y-n))/p)$. It is easily verified that this is equivalent to Theorem 4 (given G, set $F(y)=n+G(y-n)$, $y\in[n, n+1)$, and given F set $G(\tau)=F(\tau)$, $\tau\in[0, 1]$).

PROOF OF THEOREM 1.

LEMMA. *Let* (A, f_n) *be any algorithm basis and let* c *be any infinite sequence* $\{c(0), c(1), \cdots\}$ *such that* $c(0) \in P$, $c(i) \in P_0$ *for* $i > 0$. *Let* $F_k = f_{c(0)}^{-1} \cdots f_{c(k)}^{-1}(M) = f_{c(0)}^{-1} \cdots f_{c(k-1)}^{-1}(M_{c(k)})$. *Then* $f(x) = c$ *if and only if* $x \in \bigcap_0^\infty F_k$.

PROOF OF LEMMA. F_k consists exactly of those points y which correspond, under f, to sequences with at least $k+2$ entries, the first $k+1$ of which are $c(0), \cdots, c(k)$, and the lemma follows immediately from this fact. Proceeding with the theorem, let h be 1-1 onto $C(h)$, $h \sim g$, and let $h(x) = c$. If $c = \{c(0)\}$, then $g(x) = c$. If $c = \{c(0), \cdots, c(k)\}$, $k > 0$, then $x = h_{c(0)}^{-1} \cdots h_{c(k-1)}^{-1}(y)$ where $y \in I_{c(k)} - M_{c(k)}$ (note that in the definition of algorithm frame it was assumed that $I_n - M_n$ consists of at most one point; the reason for this is apparent, for if there were more than one point h could not be 1-1). Then if $w = g_{c(0)}^{-1} \cdots g_{c(k)}^{-1}(y)$, $g(w) = c$. If c is infinite, $c = \{c(0), c(1), \cdots\}$, let

$$H_k = h_{c(0)}^{-1} \cdots h_{c(k)}^{-1}(M) = h_{c(0)}^{-1} \cdots h_{c(k-1)}^{-1}(M_{c(k)}),$$

$$G_k = g_{c(0)}^{-1} \cdots g_{c(k)}^{-1}(M) = g_{c(0)}^{-1} \cdots g_{c(k-1)}^{-1}(M_{c(k)}),$$

and

$$r_k = g_{c(0)}^{-1} \cdots g_{c(k)}^{-1} h_{c(k)} \cdots h_{c(0)}.$$

Clearly, $H_{k+1} \subset H_k$, $G_{k+1} \subset G_k$, $G_k = r_k(H_k)$, and by the lemma, $x = \bigcap_0^\infty H_k$. Furthermore, since $h \sim g$, there are at most an even number of decreasing homeomorphisms in the composition of r_k, therefore each r_k is strictly increasing from the interval H_k onto the interval G_k. Also,

$$r_k(H_{k+1}) = g_{c(0)}^{-1} \cdots g_{c(k)}^{-1} h_{c(k)} \cdots h_{c(0)}[h_{c(0)}^{-1} \cdots h_{c(k)}^{-1}(M_{c(k+1)})] = G_{k+1}.$$

It follows from these facts that $\bigcap_0^\infty G_k$ is nonempty. To show this we consider three cases.

CASE 1. Each H_k is open. Then each G_k is open. Let $H_k = (a_k, b_k)$, $G_k = (\alpha_k, \beta_k)$. Since $\bigcap_0^\infty H_k$ consists of the point x, we must have that $\lim a_k = x > a_k$ for all k and $\lim b_k = x < b_k$ for all k (this also means that if $b_0 = \infty$ some b_k must be finite, and similarly, if $a_0 = -\infty$, some a_k is finite). Then there must be infinitely many indices k for which $a_k < a_{k+1}$. Let $a_k < a < a_{k+1}$. Then $\alpha_k < r_k(a) < r_k(a_{k+1}) = \alpha_{k+1}$, and therefore if $\alpha = \lim \alpha_k$, $\alpha > \alpha_k$ for all k. By the same kind of reasoning if $\beta = \lim \beta_k$, $\beta < \beta_k$ for all k. Since $\alpha \leqq \beta$, $\bigcap_0^\infty G_k = [\alpha, \beta]$, nonempty.

CASE 2. Each H_k is closed on one end and k_0 exists such that H_k is closed on the same end as H_{k_0}, say the left for $k \geqq k_0$. The G_k must have the same property. Let $H_k = [a_k, b_k)$, $G_k = [\alpha_k, \beta_k)$, $k \geqq k_0$. By the same reasoning as in Case 1 if $\beta=\lim\beta_k$, $\beta<\beta_k$ for all k, therefore $\bigcap_0^\infty G_k=\bigcap_0^\infty [\alpha_k, \beta]$ which is nonempty.

CASE 3. Each H_k is closed on one end but no k_0 as in Case 2 exists. Then it is easily seen that $\bigcap_0^\infty G_k=\bigcap_0^\infty \overline{G}_k$ which is nonempty. Since in all cases $\bigcap_0^\infty G_k$ is nonempty, there exists $y\in R$ such that $g(y)=c$, which completes the proof.

The proof of Corollary 1 is essentially contained in the analysis of finite sequences given above. Corollary 2 is immediate.

PROOF OF THEOREM 2. Let $h\sim g$ and both be 1-1 onto $C(h)=C(g)$. Let $x\in R$. The following function F is 1-1 from R onto R: if $h(x)=c$ then $y=F(x)$ if $g(y)=c$. Since each interval M_n consists exactly of those points which correspond under the algorithm to sequences containing at least two entries, the first of which is n, F maps M_n onto M_n. If $h(x)=\{c(0)\}$, then $F(x)=x$ so F maps I_n onto I_n. To see that F is strictly increasing, let $x<\hat{x}$, $h(x)=c$, $h(\hat{x})=d$. Define the length l of c as follows: if $c=\{c(0), \cdots, c(k)\}$, then $l=k$, and if c is infinite $l=\infty$. Let $\hat{l}$ be the length of d. There are two cases to consider.

CASE 1. There exists an integer $k\leqq\min(l, \hat{l})$ such that $c(i)=d(i)$, $i<k$, and $c(k)\neq d(k)$. If $k=0$, since $x\in I_{c(0)}$, $\hat{x}\in I_{d(0)}$, we must have that $I_{c(0)}$ is to the left of $I_{d(0)}$. Since $F(x)\in I_{c(0)}$, $F(\hat{x})\in I_{d(0)}$, $F(x)<F(\hat{x})$. If $k>0$ then we can write

$$\begin{aligned}
x &= \overset{-1}{h}_{c(0)} \cdots \overset{-1}{h}_{c(k-1)}(x_k) && \text{for some } x_k \in I_{c(k)},\\
\hat{x} &= \overset{-1}{h}_{c(0)} \cdots \overset{-1}{h}_{c(k-1)}(\hat{x}_k) && \text{for some } \hat{x}_k \in I_{d(k)},\\
F(x) &= \overset{-1}{g}_{c(0)} \cdots \overset{-1}{g}_{c(k-1)}(y_k) && \text{for some } y_k \in I_{c(k)},\\
F(\hat{x}) &= \overset{-1}{g}_{c(0)} \cdots \overset{-1}{g}_{c(k-1)}(\hat{y}_k) && \text{for some } \hat{y}_k \in I_{d(k)}.
\end{aligned}$$

Let $\overset{-1}{h}_{c(0)} \cdots \overset{-1}{h}_{c(k-1)}$ be increasing. Then $x_k<\hat{x}_k$, $I_{c(k)}$ is to the left of $I_{d(k)}$, $y_k<\hat{y}_k$ and therefore $F(x)<F(\hat{x})$ since $\overset{-1}{g}_{c(0)} \cdots \overset{-1}{g}_{c(k-1)}$ is also increasing. If $\overset{-1}{h}_{c(0)} \cdots \overset{-1}{h}_{c(k-1)}$ is decreasing then $x_k>\hat{x}_k$, $I_{c(k)}$ is to the right of $I_{d(k)}$, $y_k>\hat{y}_k$ and $F(x)<F(\hat{x})$ since $\overset{-1}{g}_{c(0)} \cdots \overset{-1}{g}_{c(k-1)}$ is also decreasing.

CASE 2. $\min(l, \hat{l})$ is finite and $c(i)=d(i)$, $i\leqq\min(l, \hat{l})$. For definiteness let $l<\hat{l}$. If $l=0$ then $x\in I_{c(0)}-M_{c(0)}$. Since $\hat{x}\in M_{c(0)}$, x is the left end point of $I_{c(0)}$. Then $F(x)=x<F(\hat{x})$ because $F(\hat{x})\in M_{c(0)}$. If $l>0$ we can write

$$\begin{aligned}
x &= \overset{-1}{h}_{c(0)} \cdots \overset{-1}{h}_{c(l-1)}(x_l), && x_l \in I_{c(l)} - M_{c(l)},\\
\hat{x} &= \overset{-1}{h}_{c(0)} \cdots \overset{-1}{h}_{c(l-1)}(\hat{x}_l), && \hat{x}_l \in M_{c(l)},\\
F(x) &= \overset{-1}{g}_{c(0)} \cdots \overset{-1}{g}_{c(l-1)}(x_l), && \\
F(\hat{x}) &= \overset{-1}{g}_{c(0)} \cdots \overset{-1}{g}_{c(l-1)}(\hat{y}_l), && \hat{y}_l \in M_{c(l)}.
\end{aligned}$$

If $h_{c(0)}^{-1} \cdots h_{c(l-1)}^{-1}$ is increasing, $x_l < \hat{x}_l$, therefore x_l is the left endpoint of $I_{c(l)}$, therefore $x_l < \hat{y}_l$ and $F(x) < F(\hat{x})$. The proof is straightforward if $h_{c(0)}^{-1} \cdots h_{c(l-1)}^{-1}$ is decreasing. Thus F is strictly increasing and is therefore a homeomorphism of R onto R. Let $x \in M_n$. Then $x = h_n^{-1}(r)$ and $F(x) = g_n^{-1}(s)$. But $s = F(r)$ (this follows from the fact that if $h(x) = \{n, a_1, a_2, \cdots\}$ and $x = h_n^{-1}(r)$ then $h(r) = \{a_1, a_2, \cdots\}$) therefore $F(x) = g_n^{-1}(F(h_n(x)))$ or $h_n^{-1} = F^{-1}g_n^{-1}F$. Conversely let $h_n^{-1} = F^{-1}g_n^{-1}F$ where F is an increasing homeomorphism from R onto R taking M_n onto M_n. Let $y = F(x)$. Then $h(x) = c$ if and only if $g(y) = c$, which completes the proof.

Proof of Theorem 3 and Theorem 4. Theorem 3 is obtained simply by applying Theorem 2 to this basis, using the inverse functional relation $h_n = F^{-1}g_nF$ and choosing $g_n(x) = 1/(x-n)$ (the corresponding g is the ordinary continued fraction algorithm which is well-known to be 1-1). Theorem 4 is obtained by taking $g_n(x) = p\cdot(x-n)$ (the corresponding g is the ordinary decimal expansion to the base p, which is 1-1). In both cases the functional relation implies that $F(x)-n$ is a function of $x-n$ only and therefore $F(x) - n = F(x-n)$ for $x \in [n, n+1)$.

Finally, let (A, h_n) be an algorithm basis giving rise to the function $h \in E$ and suppose h is 1-1. If $x \in R$ and c is an infinite sequence such that $h(x) = c$, there are two ways of interpreting the *continued function expansion* of x:

$$x = h_{c(0)}^{-1}(h_{c(1)}^{-1}(\cdots)).$$

The first is that for every $k \geqq 0$, $x = h_{c(0)}^{-1} \cdots h_{c(k)}^{-1}(y)$ where $h(y) = \{c(k+1), \cdots\}$. The second is that $x = \lim_{k\to\infty} h_{c(0)}^{-1} \cdots h_{c(k)}^{-1}(y)$ for all $y \in M$, which follows from the fact that $x = \bigcap_0^\infty H_k$.

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