ON THE IMBEDDABILITY OF CERTAIN COMPLEXES IN EUCLIDEAN SPACES

P. J. HILTON AND E. H. SPANIER[1]

1. Statement of results. Let $S^p \cup_f e^q$ $(0 < p < q)$ denote the complex obtained by attaching a q-cell e^q to the p-sphere S^p by means of a continuous map $f: S^{q-1} \to S^p$. Thus $S^p \cup_f e^q$ is the union of the mapping cylinder of f with a cone over S^{q-1}. In this note we consider the problem of imbedding such a complex in euclidean space or, equivalently, in a sphere. It is clear that the mapping cylinder of f can be imbedded in the join $S^{q-1} * S^p$, which is homeomorphic to S^{p+q}, and, therefore, $S^p \cup_f e^q$ can be imbedded in S^{p+q+1}. We shall give a condition which insures that this imbedding is into the lowest dimensional sphere in which it is possible to imbed $S^p \cup_f e^q$.

We say that the map $f: S^{q-1} \to S^p$ can be *S-desuspended* if there is a map $g: S^{q-2} \to S^{p-1}$ and an integer $k \geqq 0$ such that the $(k+1)$-fold suspension of g is homotopic to the k-fold suspension of f (denoted by $S^{k+1}g \simeq S^k f$). If there is no such g, then we say f cannot be S-desuspended. Our main result is:

THEOREM I. *Let $f: S^{q-1} \to S^p$ be a map which cannot be S-desuspended. Then $S^p \cup_f e^q$ can be imbedded in S^{p+q+1} but not in S^{p+q}.*

As an application of this theorem we see that if $f: S^{q-1} \to S^2$ $(q \geqq 4)$ is such that no suspension of f is null homotopic then $S^2 \cup_f e^q$ can be imbedded in S^{q+3} but not in S^{q+2}. In particular, the complex projective plane is homeomorphic to $S^2 \cup_f e^4$ where $f: S^3 \to S^2$ is a map of Hopf invariant one so it can be imbedded in S^7 but not in S^6.

In Theorem I note that if $p = q-1$ then any map $f: S^p \to S^p$ can be S-desuspended unless $p=1$ and f has degree different from 0 and ± 1. If $p=1$ and f has degree $\pm k$ where $k > 1$, then $S^1 \cup_f e^2$ has a finite nontrivial second cohomology group so if it could be imbedded in S^3 it would follow from the Alexander duality theorem that its complement in S^3 would have a finite nontrivial zero dimensional homology group, and this is impossible. Hence, Theorem I is proved in this case. Since any inessential map $f: S^{q-1} \to S^p$ can be S-desuspended, Theorem I does not apply for such maps. Therefore, to com-

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plete the proof of Theorem I we need only consider the range of values $1<p<q-1$.

The proof of Theorem I follows from an investigation of a weaker imbeddability which is better adapted to the S-category and duality therein. We say that a finite CW-complex X can be *S-imbedded* in S^n if there is a subcomplex Y of S^n and an integer $k\geqq 0$ such that S^kX and S^kY have the same homotopy type. Let $i(X)$ denote the least integer n such that X can be S-imbedded in S^n. The function $i(X)$ has been considered in [**2**; **3**]. Clearly X cannot be imbedded in a sphere of dimension $i(X)-1$.

Given a map $f: S^{q-1}\to S^p$ $(1<p<q-1)$ let $d(f)$ denote the greatest integer k such that there is a map $g: S^{q-k-1}\to S^{p-k}$ with $S^{k+r}g\simeq S^rf$ for some $r\geqq 0$. Then $0\leqq d(f)\leqq p$, and $d(f)=0$ if and only if f cannot be S-desuspended while $d(f)=p$ if and only if some suspension of f is null homotopic. In view of remarks made earlier Theorem I will follow if we can prove that if f cannot be S-desuspended then

$$i(S^p\cup_f e^q) \geqq p+q+1.$$

This is a consequence of the following:

THEOREM II. *Let $1<p<q-1$. Then for any map $f: S^{q-1}\to S^p$ we have*

$$i(S^p\cup_f e^q) = p+q+1-d(f).$$

The proof of Theorem II uses the duality in the S-category [**4**] and some results of [**1**]. The case $q=p+3$ of this theorem has been established by F. P. Peterson and N. Stein (in a forthcoming paper) using secondary cohomology operations.

2. **Proofs.** Before we proceed to the proof of Theorem II we list some results which will be used in the proof. Throughout this section we shall assume $1<p<q-1$.

LEMMA 1. *If $S^p\cup_f e^q$ is of the same homotopy type as a suspension, then f is homotopic to Sg for some map $g: S^{q-2}\to S^{p-1}$.*

LEMMA 2. *If $S^p\cup_f e^q$ and $S^p\cup_g e^q$ have the same homotopy type, then $f\simeq hgh'$ where $h: S^p\to S^p$ and $h': S^{q-1}\to S^{q-1}$ are homotopy equivalences.*

Lemma 1 is just Lemma 3.6 of [**1**], and Lemma 2 is proved in the proof of Lemma 3.6 of [**1**].

We let $X \# Y=X\times Y/X\vee Y$ denote the reduced product of X and Y. A *duality map*

$$v: S^p \# S^r \to S^{p+r}$$

as defined in [4] reduces in the special case of spheres to just a homotopy equivalence between $S^p \# S^r$ and S^{p+r}. It is a consequence of Theorem (5.9) of [4] that if

$$u: S^r \# S^p \to S^{p+r}, \qquad v: S^p \# S^r \to S^{p+r}$$

are duality maps there is an isomorphism

$$D(u, v): \{S^p, S^r\} \approx \{S^p, S^r\}$$

(where $\{S^p, S^r\}$ denotes the group of stable homotopy classes and is the direct limit of the groups $\pi_{p+k}(S^{r+k})$ as $k \to \infty$). Furthermore, if k is large enough and $f: S^{p+k} \to S^{r+k}$, $f': S^{p+k} \to S^{r+k}$ represent elements $\{f\}$, $\{f'\} \in \{S^p, S^r\}$, then $D(u, v)\{f\} = \{f'\}$ if and only if the following diagram is homotopy commutative (see (5.11) of [4])

$$\begin{array}{ccc} S^{p+k} \# S^{p+k} & \xrightarrow{1 \# f} & S^{p+k} \# S^{r+k} \\ f' \# 1 \downarrow & & \downarrow v_{k,k} \\ S^{r+k} \# S^{p+k} & \xrightarrow{u_{k,k}} & S^{p+r+2k} \end{array} .$$

The following answers a question raised on page 270 of [3].

Lemma 3. *The map $D(u, v): \{S^p, S^r\} \approx \{S^p, S^r\}$ is equal to ± 1 (where 1 denotes the identity map of $\{S^p, S^r\}$).*

Proof. Since $v_{k,k}$ and $u_{k,k}$ are homotopy equivalences and the map $T: S^{p+k} \# S^{r+k} \to S^{r+k} \# S^{p+k}$ which interchanges the two factors is also a homotopy equivalence, it follows that $v_{k,k} \simeq h u_{k,k} T$ where $h: S^{p+r+2k} \to S^{p+r+2k}$ is a homotopy equivalence (so has degree ± 1). From the homotopy commutativity of the diagram above we see that

$$u_{k,k}(f' \# 1) \simeq h u_{k,k} T(1 \# f) \simeq u_{k,k} h' T(1 \# f)$$

where h' is a homotopy equivalence of $S^{r+k} \# S^{p+k}$ with itself. Since $u_{k,k}$ is a homotopy equivalence, this implies that

$$f' \# 1 \simeq h' T(1 \# f).$$

Let T' be the map of $S^{p+k} \# S^{p+k}$ into itself which interchanges the two factors. Then $T(1 \# f) = (f \# 1) T'$ so

$$f' \# 1 \simeq h'(f \# 1) T'.$$

Since $f' \# 1 = S^{p+k} f'$, $f \# 1 = S^{p+k} f$ and h', T' are homotopy equivalences, the above equation implies that $\{f'\} = \pm \{f\}$ and completes the proof.

Proof of Theorem II. Let $g: S^{q-k-1} \to S^{p-k}$ be a map which cannot be S-desuspended and such that $S^{k+r} g \simeq S^r f$ (so $d(f) = k$). Then as we saw in the first paragraph of §1, $S^{p-k}\cup_g e^{q-k}$ can be imbedded in $S^{p+q-2k+1}$ so $S^k(S^{p-k}\cup_g e^{q-k})$ can be imbedded in $S^{p+q-k+1}$. Since $S^r(S^k g)\simeq S^r f$ it follows that $S^r(S^k(S^{p-k}\cup_g e^{q-k}))$ has the same homotopy type as $S^r(S^p\cup_f e^q)$ so $S^p\cup_f e^q$ can be S-imbedded in $S^{p+q-k+1}$ showing that $i(S^p\cup_f e^q)\leqq p+q+1-d(f)$.

To prove the opposite inequality note that if $d(f)=p$ there is no subset Y of S^q such that $S^k Y$ and $S^k(S^p\cup_f e^q)$ have the same homotopy type because such a Y would have to have $H^p(Y)$ and $H^q(Y)$ different from 0, and there is no subset of S^q like that. Therefore, if $d(f)=p$, $i(S^p\cup_f e^q)=q+1$.

Therefore, we may assume $d(f)<p$. Since $2\leqq p<q-1$ we must also have $d(f)<p-1$. Assume $i(S^p\cup_f e^q)<p+q+1-d(f)$ so $S^p\cup_f e^q$ can be S-imbedded in $S^{p+q-d(f)}$. Let X be a subcomplex of $S^{p+q-d(f)}$ such that $S^r X$ has the same homotopy type as $S^r(S^p\cup_f e^q)$. Let X' be a deformation retract of $S^{p+q-d(f)}-X$. Then X' is a $(p+q-d(f)-1)$-dual[2] of X (5.1 of [4]) so X' is a $(p+q+r-d(f)-1)$-dual of $S^r X$ and, hence, also of $S^r(S^p\cup_f e^q)$. Let $u\colon S^p \# S^{q-1}\to S^{p+q-1}$, $v\colon S^{q-1}\# S^p \to S^{p+q-1}$ be duality maps. Then $D(u,v)\{f\}=\{g\}$ where $\{g\}=\pm\{f\}$ by Lemma 3. By (6.10) of [4] $S^p\cup_f e^q$ and $S^p\cup_g e^q$ are $(p+q)$-dual. Then $S^r(S^p\cup_f e^q)$ and $S^p\cup_g e^q$ are $(p+q+r)$-dual. Since $S^{d(f)+1}X'$ is also $(p+q+r)$-dual to $S^r(S^p\cup_f e^q)$, it follows that, for some k, we have $S^{k+d(f)+1}X'$ is of the same homotopy type as $S^k(S^p\cup_g e^q)$. Now S^1X' is of the same homotopy type as $S^{p-d(f)}\cup_{g_1} e^{q-d(f)}$ for some $g_1\colon S^{q-d(f)-1}\to S^{p-d(f)}$ (because it is simply connected and has nontrivial integral homology groups only in dimensions $p-d(f)$ and $q-d(f)$, and these are infinite cyclic). By Lemma 1, $g_1\simeq Sg_1'$ for some $g_1'\colon S^{q-d(f)-2}\to S^{p-d(f)-1}$, and by Lemma 2, $S^k g\simeq\pm S^{k+d(f)}g_1$ so $S^k g\simeq\pm S^{k+d(f)+1}g_1'$. Therefore, $d(g)\geqq d(f)+1$. Since $\{g\}=\pm\{f\}$, $d(g)=d(f)$, and we have arrived at a contradiction.

References

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Cornell University and University of Birmingham,
University of California, Berkeley

[2] This is the notational usage of [4]. In terms of the definition and terminology of [3] X' is a $(p+q-d(f))$-dual of X.