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CONDITIONS IMPLYING CONTINUITY OF FUNCTIONS

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In the study of functions on certain types of spaces, the question naturally arises as to what additional conditions may imply that the functions are continuous. Several papers, mainly [**2**; **3**; **4**], have considered this problem. In this note, some further results of this type are developed.

To avoid repetition, a function f will be at least on a Hausdorff space X onto a Hausdorff space Y with additional restrictions stated as needed. Also f is compact preserving (connected) if when K is a compact (connected) subset of X, $f(K)$ is a compact (connected) subset of Y; f has closed point inverses if for each $y \in Y$, $f^{-1}(y)$ is closed and f is monotone if $f^{-1}(y)$ is connected. The rest of the terminology is standard.

In [**1**], it was shown that if X is regular, Y compact and if f is closed with closed point inverses, f is continuous. Combining this with Theorem 3.1 of [**4**], one has the result:

THEOREM 1. *If f is a closed monotone connected function on a regular space X onto a compact space Y, then f is continuous.*

It is easy to see that without the assumption that Y is compact, the conclusion need no longer hold.

THEOREM 2. *If X is locally compact, then if f is compact preserving and point inverses are closed, f is continuous.*

Consider any point $x \in X$. Since X is locally compact, x has a neighborhood U_0 with a compact closure Cl U_0. Because continuity is a local property, one need only consider f restricted to Cl U_0. On Cl U_0, f is closed and $f(\text{Cl } U_0)$ is compact. Hence the conditions of Theorem 3 of [**1**] are satisfied and f is continuous at x.

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If X is not locally compact, then f may not be continuous. The function in the first example on [**3**, p. 162] is such an instance.

DEFINITION 1. *A function f has at worst a removable discontinuity at $x \in X$ if there is a $y \in Y$ such that for each neighborhood V of y, there is a neighborhood U of x such that $f(U-[x]) \subset V$.*

If X satisfies the first axiom of countability, this definition is equivalent to Definition 3.2 of [**4**]. With this interpretation, the conditions of Theorem 3.6 of [**4**] may be relaxed somewhat.

THEOREM 3. *If X is locally connected and f is connected, then f is continuous at x_0 if and only if f has at worst a removable discontinuity at x_0.*

With only minor change, the proof given by Pervin and Levine applies here.

THEOREM 4. *If X is regular and f is a closed function with closed point inverses, then if f has a removable discontinuity at $x_0 \in X$, f is continuous at x_0.*

If x_0 is isolated in X, the result is obviously true. Assume that x_0 is nonisolated and f is not continuous at x_0. Let y be the point of Y determined by the hypothesis. Since $y \neq f(x_0)$ and $f^{-1}(y)$ is closed, a neighborhood U of x_0 exists such that $f^{-1}(y) \cap \mathrm{Cl}\, U = \varnothing$. Then $y \notin f(\mathrm{Cl}\, U)$ and because $f(\mathrm{Cl}\, U)$ is closed, a neighborhood V of y exists for which $V \cap f(\mathrm{Cl}\, U) = \varnothing$. There is a neighborhood W of x_0 such that $f(W-[x_0]) \subset V$. Since x_0 is nonisolated, $U \cap W - [x_0] \neq \varnothing$. Hence $\varnothing \neq f(W-[x_0]) \cap f(\mathrm{Cl}\, U) \subset V \cap f(\mathrm{Cl}\, U)$, a contradiction.

DEFINITION 2. *A space X will be said to have property K at a point x if for each infinite subset A having x as an accumulation point, there is a compact subset of $A \cup [x]$ which has x as an accumulation point.*

THEOREM 5. *If X has property K at x_0, then if f is compact preserving and has closed point inverses, f is continuous at x_0.*

It can be assumed that x_0 is nonisolated. Suppose f is not continuous at x_0 and that $\mathfrak{U}$ is the family of neighborhoods of x_0. Then for some neighborhood V of $f(x_0)$ and for each $U \in \mathfrak{U}$, there is an x_u such that $x_u \in U \cap f^{-1}(Y-V)$. Let $A = \{x_u \mid U \in \mathfrak{U}\}$. Then A is infinite, for x_0 is an accumulation point of A. By hypothesis, there is an infinite compact subset K of $A \cup [x_0]$. By Theorem 2, f restricted to K is continuous. However $f(K-[x_0]) \subset Y-V$ but $f(x_0) \in V$, a contradiction.[1]

[1] This is the referee's revision of the author's original proof.

THEOREM 6. *If X is locally connected with property K at each point and if f is compact preserving and connected, then f is continuous.*

It need only be shown that point inverses are closed.

Let $y_0 \in Y$ and suppose $x_0 \in \mathrm{Cl}\, f^{-1}(y_0) - f^{-1}(y_0)$. Denote the family of connected neighborhoods of x_0 by $\mathcal{C}$ and the family of neighborhoods of y_0 by $\mathfrak{U}$. Select disjoint open neighborhoods V and U_0 of $f(x_0)$ and y_0 respectively. For each $C \in \mathcal{C}$ and $U \in \mathfrak{U}$, let the point $y(U, C) \in f(C) \cap U \cap U_0 - [y_0]$ and the point $x(U, C) \in f^{-1}(y(U, C)) \cap C$. The set A of all such $x(U, C)$ is infinite and has x_0 as an accumulation point. By hypothesis, $A \cup [x_0]$ has an infinite compact subset K with x_0 as an accumulation point. Note that $x_0 \in K$. Let g denote the function f restricted to K. Then $S = g(K) - [g(x_0)] = g(K) \cap (Y - V)$ is an infinite compact set and must have an accumulation point z. If $x = g^{-1}(z)$ is isolated in K, then $K - [x]$ and hence $S - [z]$ are compact, a contradiction. Assume then that for each accumulation point of S, its inverse in K is an accumulation point of K.

Let A be the set of accumulation points of K, excluding x_0. For each $x \in A$, select disjoint open neighborhoods W_x and R_x of x and x_0 respectively. Each $K - W_x$ is compact and each $B_x = g(K - W_x) \cap S$ is a closed non-null subset of S. The family $\mathcal{B} = \{B_x \mid x \in A\}$ has the finite intersection property, for suppose the contrary. There would exist a finite number of neighborhoods $W_{x_1}, \cdots, W_{x_n}$ such that $K - [x_0] \subset \bigcup_{i=1}^{i=n} W_{x_i}$, but since for each W_{x_i}, there is a neighborhood R_{x_i} of x_0 disjoint from W_{x_i}, $\bigcap_{i=1}^{i=n} R_{x_i}$ is a neighborhood of x_0 disjoint from $\bigcup_{i=1}^{i=n} W_{x_i}$, a contradiction. Hence $\bigcap \{B_x \mid B_x \in \mathcal{B}\} \neq \emptyset$, and for each $y \in \bigcap \{B_x \mid B_x \in \mathcal{B}\}$, $t = g^{-1}(y)$ is an isolated point of K.

Let T be the set of such points t in K. Since T is open in K, for each $x \in A$ the set $K - (W_x \cup T)$ is compact and non-null. Then $\bigcap \{g(K - (W_x \cup T)) \mid x \in A\} \cap S$ is a null intersection of non-null closed subsets of a compact set, and there is a finite number of neighborhoods $W_{x_1}, \cdots, W_{x_m}$ which covers $K - (T \cup [x_0])$. Since x_0 is an accumulation point of K, T must be infinite and hence $T \cup [x_0]$ has a compact infinite subset H whose only accumulation point is x_0. Then $g(H) \cap S$ is an infinite compact subset of S and must have an accumulation point z, a contradiction since $g^{-1}(z)$ is an isolated point of K.

Thus it follows that $x_0 \in f^{-1}(y_0)$ and $f^{-1}(y_0)$ is closed. By Theorem 5, f is continuous.

Spaces satisfying the first axiom of countability obviously have property K. In conclusion there is given an example of a space in which property K holds but local countability does not.

Let $X = \bigcup_{n=-\infty}^{\infty} S_n$ be a planar set of points where for

$$n \geqq 1, \quad S_n = \{(x, y)/0 \leqq x \leqq n/(n+1),\ y = x(n+1)/n\};$$
$$n \leqq -1, \ S_n = \{(x, y)/0 \geqq x \geqq n/(1-n),\ y = x(1-n)/n\} \text{ and}$$
$$S_0 = \{(x, y)/0 = x,\ 0 \leqq y \leqq 1\}.$$

For any point $(x, y) \neq (0, 0)$, let the neighborhoods of (x, y) be defined by relativization of the Euclidean topology of the plane. At $(0, 0)$, define a base as composed of sets of the form $\bigcup_{n=-\infty}^{n=\infty} E_n$, where for each integer n, $E_n \subset S_n$ is a half-open interval with $(0, 0)$ as the endpoint. Then the first axiom of countability is not satisfied at $(0, 0)$, but the space does have property K there.

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