CEVIAN SIMPLEXES

SAHIB RAM MANDAN

Inspired by the works of Court [**3-10**] in regard to cevian triangles and tetrahedra, it is proposed to introduce their analogues in higher spaces, establish a number of new results, and observe their interesting relations with the associated quadrics and S-configurations. The knowledge of the axioms of incidence in a projective space of n dimensions, or briefly in an n-space, is assumed. The treatment is mainly synthetic and the analytic is suggested. The notations used here for intersections, joins, harmonic conjugacies, cross ratios, etc. follow Coxeter [**11**].

1. **Introduction.** a. Let M be a point and $S=(A_i)$ a simplex in an n-space with vertices at the $n+1$ general points A_i other than M; the $(n-1)$-dimensional simplex as well as the hyperplane of S opposite A_i be both denoted by its *prime face* a^i, its $(n-2)$-dimensional simplex and $(n-2)$-space opposite its edge A_iA_j by its $(n-2)$-*face* a^{ij}, etc.; but for brevity, A_iA_j be denoted by a_{ij}, the triangle $A_iA_jA_k$ as well as the plane A_ka_{ij} of S be both denoted by its *plane face* a_{ijk}, its tetrahedron $A_iA_jA_kA_l$ and the solid A_la_{ijk} by its *solid face* a_{ijkl}, etc.; $a^i\cdot A_iM=A_i'$, $Ma_{ij}\cdot a^{ij}=M^{ij}$, $Ma^{ij}\cdot a_{ij}=M_{ij}$, $Ma^{ijk}\cdot a_{ijk}=M_{ijk}$, $M^{ijk}=a^{ijk}\cdot Ma_{ijk}$, $\cdots$. $S'=(A_i')$ is then another simplex perspective to S from M. If p be an element of S or associated with S, let the corresponding one of S' be denoted by p'. Now by axioms of incidence only we have the following

LEMMA 1. *The C_2^{n+1} joins $M_{ij}M^{ij}$, the $C_3^{n+1}M_{ijk}M^{ijk}$, $\cdots$, all concur at M as the secants through M to the corresponding pairs of opposite spaces of the simplex S. For every given value of i, the n joins A_jM^{ij}, the C_2^n $M_{jk}M^{ijk}$, $\cdots$, concur at A_i'. For every given value of i, j, the $n-1$ joins A_kM^{ijk}, the $C_2^{n-1}M_{kl}M^{ijkl}$, etc. concur at M^{ij} $\cdots$. For every given value of i, j, k, l, the 4 joins A_iM_{jkl} and the 3 $M_{ij}M_{kl}$ concur at M_{ijkl}. For every given value of i, j, k, the 3 joins A_iM_{jk} concur at M_{ijk}.*

b. DEFINITION 1. The joins A_iA_i', $M^{ij}M_{ij}$, $M^{ijk}M_{ijk}$, $\cdots$, are respectively said to be the *cevians*, *bicevians*, *3-cevians*, etc. of S. S' is called the *cevian simplex* of S for M, or of M for S which is then referred to as the *anticevian simplex* of S' for M in analogy with such triangles and tetrahedra [**4-7**].

As an evident consequence we have another

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LEMMA 2. *The cevians of all orders of a simplex S through a point M lie respectively along those of its cevian as well as anticevian simplex for M.*

c. LEMMA 3. *The hyperplane m of perspectivity of a simplex S and its cevian simplex S' for a point M coincides with the "polar" of M for S or S'.*

PROOF. The 4 collinear triads of points A_iMA_i', A_jMA_j', $A_iA_j'M^{ij}$, $A_i'A_jM^{ij}$ are coplanar and form a quadrilateral q whose diagonal triangle is $M_{ij}'A_{ij}M_{ij}$ (§1a), where $A_{ij}=a_{ij}\cdot a_{ij}'=A_{ij}'$. Now by the harmonic property of q, $H(A_iA_j,\ M_{ij}A_{ij})$, $H(A_i'A_j',\ M_{ij}'A_{ij}')$. Again the *polar hyperplane* m [**14**; **15**] of M for S or S' contains the C_2^{n+1} points $A_{ij}=A_{ij}'$. Hence the result.

2. **Medial simplex.** Let G, G^i, G^{ij}, $\cdots$, G_{ijkl}, G_{ijk} be respectively the centroids of the simplex S (§1a), its prime face a^i, $(n-2)$-face a^{ij}, $\cdots$, tetrahedron a_{ijkl}, triangle a_{ijk}, and G_{ij} be the midpoint of its edge a_{ij}. A_iG^i, $G^{ij}G_{ij}$, $G^{ijk}G_{ijk}$, $\cdots$, are then respectively its *medians*, *bimedians*, 3-*medians*, etc. and (G^i) is its *medial simplex* evidently *homothetic* to it w.r.t. G with $GG^i\colon GA_i=-1\colon n$ as their homothetic ratio [**16**]. Thus by Lemma 3 we have

LEMMA 4. *The medial simplex of a given simplex is its cevian simplex for their common centroid, the hyperplane of their perspectivity being at infinity, and is therefore projectively equivalent to its cevian simplex for any other point.*

3. **Homology.** Let m be a hyperplane, M a point outside it, and k a constant. A point P is said to transform into P' in the *homology* $(M,\ m,\ k)$, if the biratio, cross ratio or anharmonic ratio $(PP',\ MQ)$ $=(PM\cdot P'Q)/(QP\cdot MP')=k$, where $Q=m\cdot MP$. If m be at infinity, this homology becomes the *homothety* $(M,\ k)$ such that P, P' are homothetic w.r.t. M with $k=MP/MP'$ as its ratio. Now the cevian simplex S' of S (§1b) for M becomes its medial, if M lies at their common centroid G (§2) and consequently m recedes to infinity. Thus we have

THEOREM 1. *A simplex in an n-space transforms into its cevian simplex S' for a point M in the homology $(M,\ m,\ -n)$, m being the hyperplane of perspectivity of S' and the given simplex S.*

4. **The converse proposition.**

THEOREM 2. *If the simplex S and its elements be as in* §1a *and C_2^{n+1} points M_{ij} be marked on its edges a_{ij}, one on each edge, such that the 3* *cevians A_jM_{ki} of each of its C_2^n plane faces a_{ijk} through a vertex A_i concur respectively at M_{ijk}, the same is true of its every triangle; the 4 cevians A_iM_{jkl} and the 3 bicevians $M_{ij}M_{kl}$ of its every tetrahedron a_{ijkl} concur respectively at M_{ijkl}; $\cdots$; the C_2^{n+1} hyperplanes $M_{ij}a^{ij}$ all CONCUR at M where also CONCUR the C_3^{n+1} $(n-2)$-spaces $M_{ijk}a^{ijk}$, the C_4^{n+1} $(n-3)$-spaces $M_{ijkl}a^{ijkl}$, and so on.*

PROOF. Let A_{ij} be the C_2^{n+1} points on the edges a_{ij} of S such that $H(A_iA_j, M_{ij}A_{ij})$. The 3 points A_{ij}, A_{jk}, A_{ki} are then collinear in the *harmonic or trilinear polar* m_{ijk} [**7**; **9**; **11**] of M_{ijk} for the triangle a_{ijk}. The C_2^n such lines lie in the hyperplane m determined by the n points A_{ij} on its edges through A_i. Thus m contains all the C_2^{n+1} points A_{ij}. Hence A_{ij}, M_{ij} form the C_2^{n+1} pairs of opposite vertices of an S-*configuration* [**14**] such that the pole M of m for S is one of the 2^n vertices of its *dual* thus satisfying the requisite conditions of the theorem by Lemmas 1 and 3.

We may project m to infinity to connect M with S as its centroid G and be at ease for the rest of the proposition as the necessary consequence.

5. **The cevian chain.** a. The homological relation $(M, m, -n)$ of the 2 simplexes S and S' (§3) discloses that we can derive either from the other, given m or M. We may also repeat the construction in either direction to construct a cevian simplex S'' of S' as well as the anticevian $'S$ of S for M or m.

DEFINITION 2. S'' may be called the *second cevian simplex* of S as well as the *third cevian* of $'S$ for M or m, and $'S$ may be referred to as the *second anticevian simplex* of S' as well as the *third anticevian* of S'' for M or m (cf. [**9**]).

b. This construction of the chain of cevian as well as anticevian simplexes of S for M may be continued indefinitely and we may note the following properties in regard to them (cf. [**9**]):

(i) The polar hyperplane m of M for every simplex, or link, of the chain, is the same, and hence the same is true of its cevians of all orders (Lemma 2).

(ii) Any 2 links of the chain are perspective from M with m as the hyperplane of their perspectivity.

(iii) Of 2 consecutive links of the chain, one is inscribed in the other and forms its cevian simplex for M.

(iv) Any link of the chain may play the role of the initial one.

(v) Such a cevian chain is determined by a simplex and either a point M or a hyperplane m.

(vi) Associated with a chain of cevian simplexes perspective from a point M is a chain of S-configurations (§4) such that they all have C_2^{n+1} vertices common in their common hyperplane m and their other vertices lie on the C_2^{n+1} common bicevians of the simplexes, concurrent at M, one vertex of each configuration on each line.

c. Following Court [9] or making repeated use of the biratio of the homology of §5a by projecting m to infinity (§3) we can prove

THEOREM 3. *If S, M, m be as in* §3, *the ith cevian simplex of the the simplex S for M is its transform in the homology* $(M, m, (-n)^i)$.

(This result provides a method to construct the cevian simplex of any order of S for M without constructing the intermediate ones.)

THEOREM 4. *The biratio of* 4 *collinear vertices of* 4 *cevian simplexes of order p, q, r, s of S for M, or that of these* 4 *links of the chain of cevian simplexes perspective from M and starting from S, is* $(t^p-t^r)(t^q-t^s)/(t^p-t^s)(t^q-t^r)$, *where* $t=-n$. *Hence the biratio of any* 4 *consecutive links is the same constant, anywhere in the chain, equal to* $(n-1)^2/(n^2-n+1)$.

6. **Cevian quadric.** a. In an n-space $n(n+3)/2$ general points always determine an $(n-1)$-quadric. Let Q be one which touches the n edges of the simplex S (§1a) through a vertex A_i at their respective points M_{ij} and passes through the other C_2^n points M_{jk}. The section of Q by a plane face a_{ijk} of S is then a conic q_{ijk} which touches its 2 edges a_{ij}, a_{ki} respectively at M_{ij}, M_{ki} and passes through M_{jk}. Now the 3 cevians A_iM_{jk} of the triangle a_{ijk} concur at M_{ijk} by Lemma 1. Then by Gergonne's theorem [8] q_{ijk} touches its third side a_{jk} at M_{jk} with M_{ijk} as its Gergonne point [10] for q_{ijk}. Thus the C_2^n edges a_{jk} of S, too, touch Q at their respective points M_{jk}. Hence we have the following

THEOREM 5. *There always exists a quadric Q tangent to the edges of a simplex S at the respective feet of its bicevians concurrent at a point M* [12].

DEFINITION 3. Q may be said to be the *cevian quadric* of S for M which may be referred to as the *pole of contact* of Q for S. When M lies at the centroid G of S, Q may be called the *ellipsoid of Steiner*, or briefly "*eSt*" of S, for then it cuts the plane faces of S in their *inscribed ellipses of Steiner* [8; 10].

b. Following Theorem 2 we may prove the converse proosition as

THEOREM 6. *If C_2^{n+1} points be marked on the edges of a simplex S in an n-space, one on each edge, such that there exist C_2^n conics, one in each of its C_2^n concurrent plane faces touching its* 3 *edges therein at their re-*

spectively marked points, the same is true of its every triangle and the C_3^{n+1} *such conics all lie on its cevian quadric* Q *for a point* M *such that the marked points are the feet of its bicevians through* M.

As an evident consequence we have the following

THEOREM 7. *If* Q *be the cevian quadric of the simplex* S *for a point* M, *and* S *along with its associated elements be as in* §1a, *the section of* Q *by a prime face* a^i *of* S *is its cevian* $(n-2)$*-quadric for* $A_i^j, \cdots,$ *and that by a plane face* a_{ijk} *of* S *is its cevian conic* q_{ijk} *for* M_{ijk}.

c. The n points M_{ij} on the n edges of the simplex S through a vertex A_i determine the polar hyperplane p^i of A_i for the quadric Q (§6a) by virtue of its construction. The n points M^{ij} (§1a) determine the hyperplane a^i of S. Now if m (§1c) be the hyperplane at infinity, M lies at the centroid G of S, M^{ij} at the centroid G^{ij} of its $(n-2)$-face a^{ij} (§2), and M_{ij} at the midpoint G_{ij} of its opposite edge a_{ij} such that $GG_{ij}: G^{ij}G = (n-1): 2$, i.e., p^i transforms into a^i in the homothety $(G, (1-n)/2)$. Thus by §3 we have the following

THEOREM 8. *If* S, M, m *be as in* §3 *and* Q *be the cevian quadric of the simplex* S *for* M, *the polar simplex of* S *for* Q *transforms into* S *in the homology* $(M, m, (1-n)/2)$. *Consequently the polar hyperplane* m *of* M *for* S *coincides with that for* Q.

COROLLARY 1. *The* $n+1$ *vertices of* S *together with* M *form a self-conjugate* $(n+2)$*-ad of points for* Q *such that the line joining any two of them is conjugate to the hyperplane containing the other* n *points for* Q *and consequently the polar* $(n-p)$*-space for* Q *of the* $(p-1)$*-space containing any* p *of them lies in the* $(n-p+1)$*-space containing the other* $n-p+2$ *points. Every one of these* $n+2$ *points is the center of perspectivity of the simplex formed of the other* $n+1$ *points and its polar for* Q [**1**, Ex. 1, p. 218; **17**].

COROLLARY 2. *The center of the "eSt" of a simplex lies at its centroid* (§6c).

d. The collinearity of the points M^{ij}, M, M_{ij} (§1a) implies that their polar hyperplanes for the quadric Q are coaxial. The polar $(n-3)$-space of M^{ij} for the $(n-3)$-quadric section of Q by the $(n-2)$-space a^{ij} of the simplex S lies in the polar hyperplane m of M for Q as that for its $(n-2)$-face a^{ij} by Theorems 7 and 8. Thus we have

THEOREM 9. *If* S, M, M_{ij}, m, a^{ij}, a_{ij} *be as in* §1 *and* Q *be the cevian quadric of the simplex* S *for* M, *the tangent hyperplane of* Q *at* M_{ij} *is* $(m\cdot a^{ij})a_{ij}$.

7. **Ring contact of cevian and polar quadrics.** a. The cevian conic q_{ijk} of the triangle a_{ijk} for M_{ijk} (§6a) is the polar conic [11] of this point for its cevian triangle $M_{ij}M_{jk}M_{ki}$ [9]. (It may be remarked here that this is not true in a solid or higher spaces, i.e., the cevian quadric of a tetrahedron or a simplex for a point M is *not* the *polar quadric* of M for its cevian tetrahedron or simplex for M.) The polar line m_{ijk} (§4) of M_{ijk} for either triangle coincides with that for q_{ijk} and therefore with that for the polar conic w_{ijk} of M_{ijk} for a_{ijk}. Now Court [9] has shown that conics like w_{ijk}, q_{ijk} *belong to a pencil of doubly tangent conics, their common chord of contact being* m_{ijk}. Again the C_3^{n+1} conics like q_{ijk} lie on the cevian quadric Q of the simplex S for M (§6a); the conics like w_{ijk}, each circumscribing the triangle a_{ijk}, are seen to lie on the polar quadric W, of M for S, circumscribing S; the C_3^{n+1} lines m_{ijk} lie in the polar hyperplane m of M for S, Q, or W. Thus follows

THEOREM 10. *The cevian and the polar quadrics Q, W of a point M for a simplex S have a "ring" contact along the polar hyperplane m of M for S* [2, p. 103].

b. *Analytically* it becomes almost obvious. For the equation of the cevian quadric Q of the simplex S for a point M referred to S may be at once put down as $\sum x_i^2 - 2\sum x_i x_j = 0$ [12], M being the unit point $(1, \cdots, 1)$ of S. It is equivalent to $4\sum x_i x_j - (\sum x_i)^2 = 0$ thus showing that Q has *ring* contact with the polar quadric $W \equiv \sum x_i x_j = 0$ of M for S.

8. **Associated S-configuration.** We have seen in §4 how the feet M_{ij} of the bicevians of the simplex S through M on its edges and their "harmonics" A_{ij} thereat form an S-configuration $(S-C)$ with M as a vertex of its dual or reciprocal $(R \cdot S-C)$. The $(S-C)$ has 2^n hyperplanes, one of them being m, containing all the A_{ij}, as the polar of M for S [14]. Let M_s be the other vertices of the $(R \cdot S-C)$ as the poles of the other hyperplanes m^s of the $(S-C)$ for their common diagonal-simplex S. Every m^s contains C_2^{n+1} points M_{ij} or A_{ij} and is determined by n such points on any n concurrent edges of S, one on each edge. Hence the hyperplanes p^i of the polar simplex of S for its cevian quadric Q (§6c) coincide with $n+1$ hyperplanes of the $(S-C)$ (cf. [8]). We can now construct the cevian quadric Q^s of S for every M_s and have

THEOREM 11. *The cevian quadrics of a simplex S in an n-space occur in sets of 2^n each such that the poles of contact of those of a set for S form the vertices of the dual of an S-configuration $(S-C)$ with S as* *their common diagonal simplex. Each quadric of the set touches the edges of S at the vertices of the $(S-C)$ other than those lying in its corresponding hyperplane. The polar of each vertex of S for every quadric of the set is a hyperplane of the $(S-C)$.*

9. **Pencil of cevian and polar quadrics.** From Theorem 10 and the homological character of the links in a chain (§5) of cevian simplexes perspective from a point M follows the following

THEOREM 12. *The cevian and the polar quadrics of a point M for the links of a chain of cevian simplexes perspective from M belong to a pencil having a ring contact along the common hyperplane of perspectivity of the chain.*

10. **Bipunctual quadric.** a. Hameed [**12**] proves *analytically* that the feet of 2 sets of bicevians, of a simplex S, concurrent respectively at 2 points K, L, on its edges lie on a quadric U. We can establish the same *synthetically* too by the method of §6a based upon the corresponding theorem for a triangle [**1**, Ex. 29, p. 53] as done by Court [**6**] for a tetrahedron and pointed out by Hameed himself [**12**]. The *harmonically conjugate* [**2**] or *harmonic* [**12**] quadric of U splits up into the pair of polar hyperplanes of K, L for S. Thus we have

THEOREM 13. *If a quadric U passes through the feet of a set of concurrent bicevians of a simplex S on its edges, U meets them again in the feet of another set of concurrent bicevians of S. If n pairs of points of U on n concurrent edges of S be "isotomic" (i.e., equidistant from their respective midpoints), all such pairs behave alike, and U then becomes the "eSt" of S in the limit when these pairs of points coincide on their respective edges* (§6a).

COROLLARY. *If K_{ij}, L_{ij} be pairs of isotomic points on the edges of a simplex S such that K_{ij} are the feet of its bicevians concurrent at K, L_{ij}, too, lie at the feet of its bicevians concurrent at L (say).*

DEFINITION 4. K, L in this corollary may be referred as a pair of *isotomic conjugate*, or briefly *isotomic*, points for S in analogy with such points for a triangle [**7**], and their polars for S as a pair of *isotomic hyperplanes* for S. The quadric like U in general may be said to be *bipunctual* for S w.r.t. the pair of points K, L in analogy with such a conic for a triangle [**1**; **13**].

b. We may thus develop the theory of the cevian quadric Q of a simplex for a point M as a limit of its bipunctual quadric U w.r.t. a pair of points K, L when K, L coincide at M [**12**].

11. **Special simplexes.** a. Following Court [3] and making use of Theorem 6 we can now prove the following

THEOREM 14. *If $S=(A_i)$ be the simplex with vertices at the centers of $n+1$ hyperspheres, in an n-space, which touch one another externally, the C_2^{n+1} points M_{ij} of their contact lie on the hypersphere (R) orthogonal to the given hyperspheres such that their C_2^{n+1} centers of similitude A_{ij} lie in a hyperplane m. The pole M of m for S is the point for which (R) is the cevian quadric of S. The C_2^{n+1} midpoints of the segments $A_{ij}M_{ij}$ lie in the "Newton hyperplane" p of the S-configuration with vertices at A_{ij}, M_{ij}. The radical hyperplane of (R) and the circumhypersphere (S) of S coincides with p* [**14**; **19**].

b. If a simplex S be *orthogonal* or *orthocentric* with orthocenter at H, the $n-1$ altitudes of its $n-1$ plane faces through an edge a_{ij} to a_{ij} concur at the foot M_{ij} on a_{ij} of its bialtitude to a_{ij} [**18**]. Hence by Theorems 6 and 13 we have the following

THEOREM 15. *There exists a quadric Q touching the edges of an orthogonal simplex S, in an n-space, at the feet thereat of its bialtitudes as its cevian quadric for its orthocenter H. The 9-point circles of its plane faces lie on its "first $n(n+1)$ point-sphere" U as its bipunctual quadric w.r.t. H and its centroid G. The radical hyperplane h of U and its circumhypersphere (S) coincides with the polar of H for S or Q as well as with that of G for the polar hypersphere (H) of S. GH is therefore a normal to h, i.e., h is normal to the "Euler line" of S* [**18**].

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INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

GENERALIZATION OF COHN-VOSSEN'S THEOREM

CHIN-SHUI HSÜ

In this paper Cohn-Vossen's Theorem [1; 2, pp. 127–133] is extended to a characterization of similarity in E^3. All surfaces here concerned are assumed to be orientable, closed, convex and of class C^3. All homeomorphisms between surfaces are assumed to be differentiable. A scalar C^2 function on a surface is harmonic if it satisfies the Laplace equation

$$\Delta(\phi) = 0$$

where Δ is the second differential operator of Beltrami.

LEMMA 1. (HOPF-BOCHNER [3; 4]). *The only harmonic function defined on a surface is constant.*

LEMMA 2. *Given two surfaces S, $\bar{S}$ and a homeomorphism $h: S \to \bar{S}$ where h is conformal, the ratio of the first fundamental forms $\rho = \bar{I}/I$ satisfies*

$$\Delta(\log \rho) = 2(K - \rho\bar{K})$$

where K, $\bar{K}$ are the Gaussian curvatures.

PROOF. Since the quantities on both sides of the equation are scalars it needs only to verify in a particular system of coordinates. For C^3 surfaces isothermal coordinates exists locally [5]. By employing such coordinates the verification is straightforward.

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