THE NUMBER OF LATTICE POINTS ON THE BOUNDARY OF A STAR BODY

HELMUT GROEMER

Let $f(X)$ be a real valued continuous function which is defined and non-negative for all points X of the Euclidean n-dimensional space. If for any real λ and all X

$$f(\lambda X) = |\lambda| f(X),$$

the set S of points satisfying

$$f(X) \leqq 1$$

is called a star body (cf. K. Mahler [1]). The function by which a star body is defined is determined uniquely. Suppose now that S is bounded. There exists a least number k such that for all X and Y

$$f(X + Y) \leqq k(f(X) + f(Y));$$

this number k is called the concavity coefficient of S. One has always $k \geqq 1$ and furthermore $k=1$ if and only if S is convex. Let us denote by $A_n(k)$ the greatest number which has the property that there exists an n-dimensional star body S with concavity coefficient k and a lattice that has $A_n(k)$ points on the boundary of S but no lattice point except O as an inner point of S.

Minkowski [2] proved

$$A_n(1) = 3^n - 1. \tag{1}$$

The method used by Minkowski to show (1) can be generalized to give estimations of $A_n(k)$ for any k. The following theorem will be proved. As usual $[a]$ is the greatest integer $\leqq a$, $\zeta(n)$ is the Riemann zeta-function and $\mu(i)$ the Möbius function.

THEOREM 1. *Define m_i by $m_i = [2k] + i$ $(i = 1, 2, \cdots)$ and put*

$$P_n(k) = \inf_{i=1,2,\cdots} m_i^n \sum_{d|m_i} \mu(d) d^{-n},$$

$$Q_n(k) = \sum_{i=1}^{\infty} \mu(i)\left(\left(2\left[\frac{k}{i}\right] + 1\right)^n - 1\right).$$

Then

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$$Q_n(k) \leqq A_n(k) \leqq P_n(k). \tag{2}$$

REMARK 1. Denoting by p_j the prime divisors of m_1 one has

$$\sum_{d\mid m_1} \mu(d)d^{-n} = \prod_{p_j\mid m_1} (1 - p_j^{-n}) < 1. \tag{3}$$

From this and (2) it follows that, if there are more than $[2k+1]^n-1$ lattice points on the boundary of a star body of concavity coefficient k, then there is at least one lattice point besides the origin which is an inner point of the star body. One sees in particular that Minkowski's result (1) is not only true for $k=1$, but for all k with $k<3/2$.

REMARK 2. $P_n(k)$ is defined as the greatest lower bound of an infinite set. It is however admissible to restrict the values of i to the set of integers i' satisfying

$$1 \leqq i' < \zeta^{1/n}(n)[2k+1] - [2k].$$

To prove this it is sufficient to show that for any i'' with $i''\geqq\zeta^{1/n}(n)[2k+1]-[2k]$ one has

$$m_{i''}^n \sum_{d\mid m_{i''}} \mu(d)d^{-n} \geqq [2k+1]^n \geqq P_n(k). \tag{4}$$

$[2k+1]^n \geqq P_n(k)$ follows from (3), and denoting the prime divisors of m_i by p_j one gets because of

$$\prod_{p_j\mid m_{i''}} (1 - p_j^{-n}) > \zeta^{-1}(n)$$

$$m_{i''}^n \sum_{d\mid m_{i''}} \mu(d)d^{-n} \geqq [2k+1]^n\zeta(n) \prod_{p_j\mid m_{i''}} (1 - p_j^{-n}) > [2k+1]^n$$

which implies the first inequality of (4).

For large k, (2) can not be essentially improved since one has

THEOREM 2. *If k tends to infinity*

$$A_n(k) = 2^n\zeta^{-1}(n)k^n + o(k^n), \tag{5}$$

$$P_n(k) = 2^n\zeta^{-1}(n)k^n + o(k^n), \tag{6}$$

$$Q_n(k) = 2^n\zeta^{-1}(n)k^n + o(k^n). \tag{7}$$

PROOF OF THEOREM 1. We prove first that $A_n(k)$ is a nowhere decreasing function of k. Let us denote by S a star body with concavity coefficient k, diameter D, and with $A_n(k)$ lattice points, say P_i $(i=1, 2, \cdots, A_n(k))$, on its boundary. By g we denote a half line which contains no P_i and ends at O, by r a positive number with the property that all P_i have a distance not less than r from g, and by $G(u)$ a cone with an $(n-1)$-dimensional sphere of radius r for base. Suppose that the axis of the cone is g and that the vertex and the base of $G(u)$ are at the distances u and $u+D$ from 0, respectively. Let $G'(u)$ be the reflection of $G(u)$ in O.

Consider now the set $S(u)=S-(G(u)\cup G'(u))$. $S(u)$ is obviously a star body and has the same number of lattice points on the boundary as S. The concavity coefficient of $S(u)$, say $k(u)$, is k if $u\geqq D$ and tends to infinity if u tends to 0. Since $f(X)$ is continuous it is easily seen that $k(u)$ is also a continuous function of u. For any k' with $k'\geqq k$ one can therefore find a number u' such that $k'=k(u')$. It follows that $A_n(k')\geqq A_n(k)$ if $k'\geqq k$.

To prove the right hand side of (2) one has to show that for $i=1, 2, \cdots$

$$A_n(k) \leqq m_i^n \sum_{d|m_i} \mu(d)d^{-n}, \tag{8}$$

where $m_i=[2k]+i$. If i is fixed one can find a $\bar{k}\geqq k$ such that

$$m_i = [2k] + i = [2\bar{k}] + 1 = \bar{m}_1 \text{ (say)}$$

and (8) follows because of $A_n(k)\leqq A_n(\bar{k})$ if one can prove that

$$A_n(\bar{k}) \leqq \bar{m}_1^n \sum_{d|\bar{m}_1} \mu(d)d^{-n}.$$

Instead of proving (8) for a fixed k and all i it is therefore sufficient to prove (8) for $i=1$ and all k $(k\geqq 1)$.

Let us write $A_n(k)=A$, $P_n(k)=P$, $Q_n(k)=Q$ and $m_1=m$. Suppose that $e_1, e_2, \cdots, e_n$ is a basis of a lattice Λ. Any $R\in\Lambda$ can be written uniquely in the form $R=R_m+mR'_m$ where

$$R_m = \sum_{i=1}^{n} x_ie_i, \qquad R'_m = \sum_{i=1}^{n} y_ie_i,$$

with integers x_i, y_i that satisfy

$$0 \leqq x_i < m.$$

There are m^n possibilities for the R_m. Denote by (R_m, m) the greatest common divisor of $x_1, x_2, \cdots, x_n, m$; by p_i the prime divisors of m and by C the number of R_m for which $(R_m, m)=1$. Counting all R_m with $p_i|(R_m, m)$, $p_ip_j|(R_m, m)$ and so on, one gets

$$C = m^n - \sum\left(\frac{m}{p_i}\right)^n + \sum\left(\frac{m}{p_ip_j}\right)^n - \cdots = m^n \sum_{d|m} \mu(d)d^{-n}. \tag{9}$$

If there are more than C lattice points on the boundary of a star body S with concavity coefficient k, then either for at least one of these points, say R, $(R_m, m) > 1$ or there are two points R, $\overline{R}$ on S for which $R_m = \overline{R}_m$. In the first case, the lattice point $R/(R_m, m)$ is an inner point of S. In the second case for the lattice point $R'_m - \overline{R}'_m = (R - \overline{R})/m$ one has

$$f\left(\frac{R - \overline{R}}{m}\right) \leqq \frac{k}{m}\,(f(R) + f(\overline{R})) = \frac{2k}{[2k + 1]} < 1,$$

so that $\overline{R}_m - \overline{R}'_m$ is an inner point of S. It follows that

$$A \leqq C$$

which shows (8) for $i = 1$ and all k. As already remarked, this proves the right hand side of (2). For the proof of the other side of the inequality (2), consider the closed n-dimensional cube of side length $2k$ with center at the origin and faces perpendicular to the coordinate axes. Denote this cube by $W(k)$. Let Λ be the lattice which consists of all points with integers as coordinates. The greatest common divisor of the coordinates of a lattice point P will be denoted by (P). Counting all points of Λ contained in $W(k)$ for which $p_i \mid (P)$, $p_i p_j \mid (P)$ and so on, where p_i runs now over all prime numbers, one finds for the number of primitive lattice points in $W(k)$ i.e. lattice points P with $P \in W(k)$ and $(P) = 1$,

$$(2[k] + 1)^n - 1 - \sum\left(\left(2\left[\frac{k}{p_i}\right] + 1\right)^n - 1\right) + \sum\left(\left(2\left[\frac{k}{p_i p_j}\right] + 1\right)^n - 1\right) - \cdots.$$

This expression is obviously the same as Q. Now let S be defined as follows. Corresponding to any primitive point P, construct a cone G_P with vertex at P and for base an $(n-1)$-dimensional sphere which is contained in $W(1)$ and has a radius so small that $G_P \cap \Lambda = P$. Further let G_{-P} be the reflection of G_P in the origin. Denote by S the union of all G_P and $W(1)$. By construction S is a star body that contains Q lattice points on its boundary and none except O inside. It only remains to show that the concavity coefficient of S is not greater than k, or what means the same that for all X and Y

$$f\left(\frac{X + Y}{2}\right) \leqq k\,\frac{f(X) + f(Y)}{2} \tag{10}$$

where $f(X)$ is the function which defines S. If X has the coordinates x_i $(i=1, 2, \cdots, n)$ we define a function $g(X)$ by

$$g(X) = \max_{i=1,2,\cdots,n} |x_i|.$$

Because of $W(1) \subset S \subset W(k)$ one has

$$\frac{1}{k} g(X) \leqq f(X) \leqq g(X). \tag{11}$$

Multiplying inequality (10) by a suitable positive constant justifies the assumption that $X+Y/2$ is on the boundary of $W(k)$. Under this assumption one has

$$g\left(\frac{X+Y}{2}\right) \leqq k. \tag{12}$$

Because of symmetry of the line which connects X and Y with respect to a plane which contains a face of $W(k)$ and the point $(X+Y)/2$ one gets

$$g(X) \geqq k+\gamma, \qquad g(Y) \geqq k-\gamma, \tag{13}$$

where $|\gamma|$ is the distance of X from the plane. (11), (12) and (13) give (10).

Proof of Theorem 2. (5) is a consequence of (6) and (7). Since $Q_n(k)$ is the number of primitive lattice points in $W(k)$, (7) is a special case of a well known theorem (cf. C. A. Rogers [3]). It is also easy to derive (7) directly from the sum representation which has been used to define $Q_n(k)$. We have now to prove (6). Corresponding to a number N one can find for all sufficiently large k an integer r such that

$$2k < r \prod_{p_i|N} p_i \quad \text{and} \quad 2^n k^n = \left(r \prod_{p_i|N} p_i\right)^n + o(k^n).$$

Choosing N so great that for a given ϵ, $|\zeta^{-1}(n) - \prod_{p_i|N}(1-p_i^{-n})| < \epsilon$ one obtains

$$\begin{aligned}\inf_{i=1,2,\dots} m_i^n \prod_{p_j|m_i} (1-p_j^{-n}) &\leqq \left(r \prod_{p^i|N} p_i\right)^n \prod_{p_i|N}(1-p_i^{-n}) \\ &= (2^n k^n + o(k^n))(\zeta^{-1}(n)+\epsilon)\end{aligned}$$

and so, because of

$$P_n(k) = \inf_{i=1,2,\dots} m_i^n \sum_{d|m_i} \mu(d) d^{-n} = \inf_{i=1,2,\dots} m_i^n \prod_{p_j|m_i}(1-p_j^{-n}),$$

$$k^{-n}P_n(k) \leqq 2^n\zeta^{-1}(n) + 2^n\epsilon + o(1).$$

Since this is true for any ϵ, one has

$$k^{-n}P_n(k) \leqq 2^n\zeta^{-1}(n) + o(1),$$

which gives together with (2) and (7) the relation (6).

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OREGON STATE COLLEGE

ON THE REPRESENTATION OF SEQUENCES AS FOURIER COEFFICIENTS

P. G. ROONEY

If $F \in L_p(-\infty, \infty)$, where $1 < p \leqq 2$, then F has a Fourier transform $G \in L_q(-\infty, \infty)$, where p and q are connected, now and henceforth, by the relation

$$p^{-1} + q^{-1} = 1. \tag{1}$$

However, except in the case $p = 2$, the collection of transforms does not cover $L_q(-\infty, \infty)$. In a recent paper [3], we gave a characterization of this collection of Fourier transforms.

A similar situation exists in the theory of Fourier series. If $F \in L_p(0, 2\pi)$, where $1 < p \leqq 2$, and $c = \{c_n\}$ is the Fourier sequence of F, that is

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{-int}dt, \tag{2}$$

then [5, §9.1], $c \in l_q$. That is

$$\sum_{n=-\infty}^{\infty} |c_n|^q < \infty.$$

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