THE WEIERSTRASS CONDITION FOR MULTIPLE INTEGRAL VARIATIONAL PROBLEMS INVOLVING HIGHER DERIVATIVES[1]

LAWRENCE M. GRAVES

1. **Introduction.** We consider the problem of minimizing a multiple integral

$$I = \int_G f(x, z, Dz)dx = \int \cdots \int f(x, z, Dz)dx_1 \cdots dx_n,$$

where $x = (x_1, \cdots, x_n)$, $z = (z_1, \cdots, z_q)$, z is a function of x, and Dz_k denotes the various partial derivatives of z_k with respect to the x_j up to order ν_k. When it is necessary to be more explicit, we shall let i denote an n-dimensional vector with nonnegative integer coordinates, and write

$$D^i = \prod_{j=1}^{n} D_{x_j}^{i_j}.$$

We set $|i| = \sum_j i_j$, and if $|i| = \nu_k$, denote $D^i z_k$ by p_k^i. These are the derivatives of z_k of the highest order that appear.

It is supposed that each z_k and its derivatives up to order $\nu_k - 1$ are continuous on a fixed domain G and take prescribed boundary values on the boundary G^* of G, and that the derivatives of z_k of order ν_k are piecewise continuous. We assume that the integrand f is continuous and has continuous partial derivatives with respect to the arguments p_k^i, for points (x, z, Dz) interior to a domain T. In the Weierstrass $\mathcal{E}$-function, only the arguments p_k^i are varied. Hence we shall define $Dz + P$ by the formula

$$\begin{aligned} (Dz + P)_k^i &= D^i z_k \qquad \text{for } |i| < \nu_k \\ &= D^i z_k + P_k^i \qquad \text{for } |i| = \nu_k, \end{aligned}$$

and assume for simplicity that the domain T is such that $(x, z, Dz+P)$ is in T whenever (x, z, Dz) is in T. Then we define

$$\mathcal{E}(x, z, Dz, Dz + P) = f(x, z, Dz + P) - f(x, z, Dz) - \sum_{i,k} P_k^i f_{p_k^i}(x, z, Dz),$$

Received by the editors November 23, 1959.

[1] This work was supported by the Office of Ordnance Research under Contract No. DA-11-022-ORD-1833.

where the summation index k runs from 1 to q, and i varies over the set $|i| = \nu_k$.

We shall show that if I is a minimum then

$$\mathcal{E}(x, z, Dz, Dz + P) \geqq 0$$

whenever P has the form

$$P_k^i = C_k \prod_{j=1}^{n} (\alpha_j)^{i_j}, \tag{1}$$

where C_k and α_j are arbitrary.

We may restrict attention to a point x of G near which all derivatives of z which appear are continuous, and consider only variations ζ of z which vanish outside a neighborhood of x. Then if we put

$$\bar{f}(x, \zeta, D\zeta) = f(x, z + \zeta, Dz + D\zeta) - f(x, z, Dz),$$

$$\bar{I}(\zeta) = \int \bar{f} dx,$$

we see that in (x, ζ)-space, the minimizing manifold is $\zeta = 0$, and $\bar{f}(x, 0, 0) = 0$. By a translation we may also suppose that the point x under consideration is the origin. We replace ζ by z and $\bar{f}$ by f, and understand in the proofs that any argument of f or its partial derivatives which is not written is zero. In §2 we give the proof for the case $q = 2$, $\nu_1 = 1$, $\nu_2 = 2$, and in §3 treat the general case. The method of proof is an extension of that given by the author for the case when only first derivatives appear.[2]

2. **A special case.** We consider here an integrand

$$f(x, z_1, z_2, Dz_1, Dz_2, D^2 z_2),$$

where $D^2 z_2$ stands for all the second derivatives $D_{x_j} D_{x_m} z_2$, and no derivatives appear which are of higher order than those indicated. Let

$$L_0 = \sum_{j=1}^{n} \alpha_j x_j,$$

where for convenience α is chosen as a unit vector, and for a small $b > 0$ and $|x| \leqq b$ let

$$\phi = [1 + L_0^2 - |x|^2]^{1/2} - [1 - b^2]^{1/2},$$

where $|x|$ denotes the Euclidean length of the vector x. (Note that

[2] See Duke Math. J. vol. 5 (1939) pp. 656–660.

$|i|$ was defined differently.) Then ϕ has bounded partial derivatives of all orders, and ϕ and its first partial derivatives approach zero uniformly with b. Let $1=\epsilon_0>\epsilon_1>\epsilon_2>\epsilon_3=0$, and set

$$L_1 = L_0 + (\epsilon_1 - 1)\phi,$$
$$L_2 = L_0 + (\epsilon_2 - 1)\phi,$$
$$L_3 = L_0 - \phi.$$

Then the loci $L_\beta=0$, $(\beta=0, 1, 2, 3)$, bound three adjacent domains R_0, R_1, R_2 in x-space, defined by

$$R_\beta = [x \mid L_{\beta+1} < 0 < L_\beta]. \tag{2}$$

If $V=V(b)$ denotes the volume of $R=R_0+R_1+R_2$, then the volume of R_β is $V_\beta=(\epsilon_\beta-\epsilon_{\beta+1})V$, as is readily verified by considering the special case $\alpha_1=1$, $\alpha_j=0$ for $j>1$. Also V tends to zero with b.

The variations of the minimizing manifold $z=0$ are constructed as follows. Let A_{10}, A_{20} be arbitrary constants, and let A_{11}, A_{21}, A_{22} denote functions of ϵ_1, ϵ_2, to be determined. Set

$$\begin{aligned} z_1 &= A_{10}L_0 && \text{on } R_0,\\ &= A_{10}L_0 + A_{11}L_1 && \text{on } R_1+R_2,\\ z_2 &= A_{20}L_0^2 && \text{on } R_0,\\ &= A_{20}L_0^2 + A_{21}L_1^2 && \text{on } R_1,\\ &= A_{20}L_0^2 + A_{21}L_1^2 + A_{22}L_2^2 && \text{on } R_2,\\ z_1 &= z_2 = 0 \text{ outside } R. \end{aligned}$$

Then z_1 is continuous except possibly along $L_3=0$, and z_2 and its first partial derivatives are continuous except possibly along $L_3=0$. Sufficient conditions for the required continuity along $L_3=0$ are

$$\begin{aligned} A_{10} + \epsilon_1 A_{11} &= 0,\\ A_{20} + \epsilon_1^2 A_{21} + \epsilon_2^2 A_{22} &= 0,\\ A_{20} + \epsilon_1 A_{21} + \epsilon_2 A_{22} &= 0, \end{aligned} \tag{3}$$

since $L_0=\phi$, $L_1=\epsilon_1\phi$, $L_2=\epsilon_2\phi$ on $L_3=0$.

Now when b tends to zero, so do ϕ, each L_β, z_1, z_2, and each $D_{x_j}\phi$, and hence

$$D_{x_j}L_\beta \to D_{x_j}L_0 = \alpha_j,$$
$$D_{x_j}L_\beta^2 \to 0,$$

$$
\begin{aligned}
D_{x_j}D_{x_m}L_\beta^2 &\to 2\alpha_j\alpha_m,\\
D_{x_j}z_1 &\to A_{10}\alpha_j \qquad \text{on } R_0,\\
&\to (A_{10}+A_{11})\alpha_j \qquad \text{on } R_1+R_2,\\
D_{x_j}z_2 &\to 0,\\
D_{x_j}D_{x_m}z_2 &\to 2A_{20}\alpha_j\alpha_m \qquad \text{on } R_0,\\
&\to 2(A_{20}+A_{21})\alpha_j\alpha_m \qquad \text{on } R_1,\\
&\to 2(A_{20}+A_{21}+A_{22})\alpha_j\alpha_m \qquad \text{on } R_2.
\end{aligned}
$$

Since

$$\frac{I}{V}=\frac{1-\epsilon_1}{V_0}\int_{R_0} f dx+\frac{\epsilon_1-\epsilon_2}{V_1}\int_{R_1} f dx+\frac{\epsilon_2}{V_2}\int_{R_2} f dx,$$

we find from $I(0)=0=$ minimum of $I(b)$ that

$$
\begin{aligned}
0 \leqq\ & (1-\epsilon_1)f(A_{10}\alpha_j,\ 2A_{20}\alpha_j\alpha_m)\\
&+(\epsilon_1-\epsilon_2)f[(A_{10}+A_{11})\alpha_j,\ 2(A_{20}+A_{21})\alpha_j\alpha_m]\\
&+\epsilon_2 f[(A_{10}+A_{11})\alpha_j,\ 2(A_{20}+A_{21}+A_{22})\alpha_j\alpha_m].^{3}
\end{aligned}
\tag{4}
$$

This inequality may be regarded as a generalized form of the Weierstrass condition, in which no partial derivatives of the integrand f appear. We obtain the ordinary form of the condition by dividing by $(1-\epsilon_1)$ and letting ϵ_1 tend to one. In order to evaluate this limit we need the derivatives A'_{11}, A'_{21} and A'_{22} of A_{11}, A_{21} and A_{22} with respect to ϵ_1 at $\epsilon_1=1$. Let M_β be the cofactor of s_β in the determinant

$$\begin{vmatrix} s_0 & s_1 & s_2\\ 1 & \epsilon_1 & \epsilon_2\\ 1 & \epsilon_1^2 & \epsilon_2^2 \end{vmatrix}.$$

Then from the equations (3),

$$A_{21}=A_{20}M_1/M_0, \qquad A_{22}=A_{20}M_2/M_0.$$

Also at $\epsilon_1=1$, $M_1=-M_0$, $M_2=0$, and

$$\frac{\partial}{\partial\epsilon_1}\frac{M_1}{M_0}=\frac{M'_0}{M_0}, \qquad \frac{\partial}{\partial\epsilon_1}\frac{M_2}{M_0}=\frac{M'_2}{M_0},$$

$$
\begin{aligned}
A'_{21}&=A_{20}M'_0/M_0, & A'_{22}&=A_{20}M'_2/M_0, & A'_{11}&=A_{10},\\
A_{10}+A_{11}&=0 & A_{20}+A_{21}&=0, & A_{20}+A_{21}+A_{22}&=0.
\end{aligned}
$$

[3] Here the arguments x, z_1, z_2, Dz_2 of f, which are all zero, have been omitted.

In this way we obtain from (4) the inequality

$$0 \leqq f(A_{10}\alpha_j,\ 2A_{20}\alpha_j\alpha_m)$$
$$+ (\epsilon_2 - 1)\left[\sum_j f_{p_{1j}} A_{10}\alpha_j + \sum_{j,m} f_{p_{2jm}} 2A_{20}M_0'\alpha_j\alpha_m/M_0\right]$$
$$- \epsilon_2\left[\sum_j f_{p_{1j}} A_{10}\alpha_j + \sum_{j,m} f_{p_{2jm}} 2A_{20}(M_0' + M_2')\alpha_j\alpha_m/M_0\right].$$

Now at $\epsilon_1 = 1$, $(1-\epsilon_2)M_0' + \epsilon_2(M_0' + M_2') = M_0$, so this becomes

$$0 \leqq f(A_{10}\alpha_j,\ 2A_{20}\alpha_j\alpha_m) - \sum_j f_{p_{1j}} A_{10}\alpha_j - \sum_{j,m} f_{p_{2jm}} 2A_{20}\alpha_j\alpha_m$$

or

$$0 \leqq f(p_1, p_2) - \sum_j p_{1j} f_{p_{1j}} - \sum_{j,m} p_{2jm} f_{p_{2jm}},$$

where $p_{1j} = A_{10}\alpha_j$, $p_{2jm} = 2A_{20}\alpha_j\alpha_m$, and the arguments of the partial derivatives of f are those along the minimizing manifold $z_1 = z_2 = 0$.

3. **The general case.** We let μ denote the maximum ν_k, and select ϵ_β satisfying

$$1 = \epsilon_0 > \epsilon_1 > \cdots > \epsilon_\mu > \epsilon_{\mu+1} = 0.$$

With L_0 and ϕ chosen as in §2, we set

$$L_\beta = L_0 + (\epsilon_\beta - 1)\phi.$$

There are now $\mu+1$ domains R_β defined by (2), and the domain R which is their union is defined by the inequality $0 < L_0 < \phi$. On R_β we set

$$z_k = \sum_{\sigma=0}^{\lambda} A_{k\sigma} L_\sigma^{\nu_k},$$

where λ is the lesser of β and ν_k, A_{k0} is arbitrary, and the remaining $A_{k\sigma}$ are to be determined as multiples of A_{k0}. Outside of R we set $z_k = 0$. The functions z_k and their partial derivatives up to order $\nu_k - 1$ are obviously continuous along the manifolds $L_\beta = 0$ for $\beta = 0, 1, \cdots, \mu$. To assure the required continuity along the manifold $L_{\mu+1} = 0$, it is sufficient to require that the $A_{k\sigma}$ satisfy the equations

$$\sum_{\sigma=0}^{\nu_k} A_{k\sigma}\epsilon_\sigma^\rho = 0, \qquad \rho = 1, \cdots, \nu_k,$$

as may be verified by observing that when $L_{\mu+1}=0$, $L_\sigma=\epsilon_\sigma\phi$, $DL_0=\alpha$, $DL_\sigma=DL_0+(\epsilon_\sigma-1)D\phi$, $D^2L_\sigma=(\epsilon_\sigma-1)D^2\phi$, etc. Hence we take

$$A_{k\sigma} = A_{k0}M_{k\sigma}/M_{k0},$$

where $M_{k\sigma}$ is the cofactor of s_σ in the determinant

$$\Delta = \begin{vmatrix} s_0 & s_1 & s_2 \cdots s_{\nu_k} \\ 1 & \epsilon_1 & \epsilon_2 \cdots \epsilon_{\nu_k} \\ 1 & \epsilon_1^2 & \epsilon_2^2 \cdots\cdots \\ \cdots & \cdots & \cdots\cdots \\ 1 & \epsilon_1^{\nu_k} & \cdots\cdots \epsilon_{\nu_k}^{\nu_k} \end{vmatrix}.$$

When b tends to zero, each z_k, with its derivatives up to order ν_k-1, tends to zero, and for $|i|=\nu_k$, D^iz_k tends to $p^i_{k\beta}$ on R_β, where

$$(5)\qquad p^i_{k\beta} = \nu_k! \prod_{j=1}^{n} (\alpha_j)^{i_j} \sum_{\sigma=0}^{\lambda} A_{k\sigma} = B^i_k \sum_{\sigma=0}^{\lambda} A_{k\sigma} = B^i_k A_{k0} \sum_{\sigma=0}^{\lambda} M_{k\sigma}/M_{k0}.$$

As in §2 we may write

$$\frac{I}{V} = \sum_{\beta=0}^{\mu} \frac{\epsilon_\beta - \epsilon_{\beta+1}}{V_\beta} \int_{R_\beta} f dx,$$

and derive as before the inequality

$$(6)\qquad 0 \leqq \sum_{\beta=0}^{\mu} (\epsilon_\beta - \epsilon_{\beta+1}) f(p_\beta).$$

Then we divide (6) by $(1-\epsilon_1)$ and let ϵ_1 tend to unity. In order to evaluate the result we observe the following relations.

If in the determinant Δ we put $s_0=s_1=1$, $s_\sigma=\epsilon_\sigma$ for $\sigma>1$, we find by differentiating the expansion of Δ on the first row that

$$(7)\qquad \frac{\partial \Delta}{\partial \epsilon_1} = \frac{\partial M_{k0}}{\partial \epsilon_1} + \sum_{\sigma=2}^{\nu_k} \epsilon_\sigma \frac{\partial M_{k\sigma}}{\partial \epsilon_1}.$$

By first subtracting the second row from the first and then differentiating, we find

$$(8)\qquad \frac{\partial \Delta}{\partial \epsilon_1} = - M_{k1}.$$

At $\epsilon_1=1$, we find

$$(9)\qquad M_{k1} = - M_{k0}, \qquad M_{k\beta} = 0 \text{ for } \beta > 1,$$

$$\frac{\partial}{\partial \epsilon_1} \frac{M_{k1}}{M_{k0}} = \frac{1}{M_{k0}} \frac{\partial M_{k0}}{\partial \epsilon_1},$$

$$\frac{\partial}{\partial \epsilon_1} \frac{M_{k\beta}}{M_{k0}} = \frac{1}{M_{k0}} \frac{\partial M_{k\beta}}{\partial \epsilon_1} \qquad \text{for } \beta > 1.$$

$$\frac{\partial}{\partial \epsilon_1} \overset{i}{p}_{k\beta} = \overset{i}{B}_k \frac{A_{k0}}{M_{k0}} \frac{\partial}{\partial \epsilon_1} \left[M_{k0} + \sum_{\sigma=2}^{\lambda} M_{k\sigma} \right], \tag{10}$$

$$\begin{aligned} \sum_{\beta=1}^{\mu} (\epsilon_\beta - \epsilon_{\beta+1}) \frac{\partial}{\partial \epsilon_1} \left[M_{k0} + \sum_{\sigma=2}^{\lambda} M_{k\sigma} \right] \\ = \frac{\partial}{\partial \epsilon_1} \left[M_{k0} + \sum_{\sigma=2}^{\nu_k} \epsilon_\sigma M_{k\sigma} \right] = M_{k0}. \end{aligned} \tag{11}$$

The last equality follows from (7), (8) and (9). So from (6) we have

$$\begin{aligned} 0 &\leqq f(\overset{i}{B}_k A_{k0}) - \sum_{\beta=1}^{\mu} (\epsilon_\beta - \epsilon_{\beta+1}) \sum_{k,i} f_{p_k^i} \frac{\partial}{\partial \epsilon_1} \overset{i}{p}_{k\beta} \\ &= f(\overset{i}{B}_k A_{k0}) - \sum_{k,i} \overset{i}{B}_k A_{k0} f_{p_k^i}(0), \end{aligned}$$

with the help of (10) and (11). Since by (5)

$$\overset{i}{B}_k = \nu_k! \prod_{j=1}^{n} (\alpha_j)^{i_j},$$

and since we have assumed $f(0) = 0$, the result has the form given in §1.

THE UNIVERSITY OF CHICAGO