FIXED-POINT THEOREMS FOR ARCWISE CONNECTED CONTINUA[1]

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L. E. Ward, Jr., recently proved in [4] a fixed-point theorem for certain arcwise connected spaces that generalizes a theorem of mine—Theorem 2, below—and Borsuk's theorem [1] that an arcwise connected hereditarily unicoherent metric curve has the fixed-point property. His argument provides a proof of my result, but not of Borsuk's. That Borsuk's class of continua is contained in his follows from Borsuk's result only.

In this note I give a new sufficient condition for the fixed-point property that implies Borsuk's result, and that follows from my theorem and so from Ward's. I also give an example of an arcwise connected continuum that contains no simple closed curve but that does not have the fixed-point property, and prove a fixed-point theorem for a quite special class of contractible continua.

THEOREM 1. *Let M be an arcwise connected compact Hausdorff space that does not have the fixed-point property. Then M contains either* (1) *a continuum N_1 for which there is a map $f: N_1 \to S^1$ which is onto and such that no closed proper subset of N_1 is mapped by f onto S^1, and which is such that at most one point-inverse is nondegenerate, that one being connected*; *or* (2) *a continuum N_2 that contains a subset R that is the one-to-one continuous image of a half-open interval and that is dense in N_2, but that has no interior relative to N_2*; *or* (3) *a continuum N_3 that is the union of a set R that is the continuous one-to-one image of a half-open interval, and a continuum B, and for which there is a map $f: N_3 \to K$, K being the union of the circles $x^2+y^2=(2/n)y$, $n=1, 2, 3, \cdots$, such that f is one-to-one on N_3-B, such that $f(B)=(0, 0)$, and such that no closed proper subset of N_3 is mapped by f onto K.*

We will see that this is a consequence of an earlier fixed-point theorem of the author's, proved in [5, p. 493]:

THEOREM 2. *Let M be an arcwise connected Hausdorff space which is such that every monotone increasing sequence of arcs is contained in an arc. Then M has the fixed-point property.*

Note that compactness is not required in Theorem 2.

Presented to the Society, January 29, 1960; received by the editors February 4, 1960.

[1] Part of the work on this paper was done under a grant by the National Science Foundation.

PROOF OF THEOREM 1. The proof is a straightforward analysis of the possible ways the hypothesis of Theorem 2 can fail in a compact space. Let $A_1, A_2, A_3, \cdots$ be a monotone increasing sequence of arcs that is not contained in an arc. Let x be any non-end point of A_1. Then, for each n, x divides A_n into two arcs A_n' and A_n'', the primes being chosen so that, for each n, A_n' is contained in A_{n+1}'. If M contains a simple closed curve, then that is a continuum of type 1. Suppose that M contains no simple closed curve. Then at least one of the two monotone increasing sequences $\{A_n'\}$, $\{A_n''\}$ does not lie in an arc. Hence there is no loss in assuming that $\{A_n\}$ itself is a sequence of arcs all having a common end point, a. There is also no loss in assuming that $A_{n+1}-A_n$ is never empty. Let B denote the set lim sup $\mathrm{Cl}(A_{n+1}-A_n)$. An argument of a familiar type shows that B is connected. For if B were not, there exist two disjoint open sets U and V, covering B, and each intersecting B. From some integer k on, each set $\mathrm{Cl}(A_{n+1}-A_n)$ lies either in U or in V. But $\mathrm{Cl}(A_{n+1}-A_n)$ and $\mathrm{Cl}(A_{n+2}-A_{n+1})$ intersect. Induction shows that for $n>k$, either all the sets $\mathrm{Cl}(A_{n+1}-A_n)$ lie in U, or all lie in V. This gives a contradiction.

The set $\bigcup_n A_n=R$ is the one-to-one continuous image of a half-open interval. There are three possible relations between R and B: (1) The sets R and B are disjoint. Then there is an arc xy from some point x of B to some point y of R, such that $x=xy\cap B$ and $y=xy\cap R$. The point y separates R into two connected sets, R' and R'', where $R''\cup y$ is an arc from a to y, and $R'\cup y$ is again the one-to-one continuous image of a half-open interval. (In this case, it is actually a homeomorph of such an interval.) Let $N_1=xy\cup R'\cup B$. The collection consisting of the set B and of the individual points of N_1-B is upper semi-continuous and defines a map $f: N_1\rightarrow S^1$ satisfying the conditions of part (1) of the conclusion of the theorem. (2) The sets R and B intersect, but some arc au of R contains $R\cap B$. (It may actually happen that $R\cap B$ consists of just two points.) Let R' be the set $R-au$, and let $N_1=R'\cup B$. Then in the same way as above, we have the desired map $f: N_1\rightarrow S^1$. (3) There is an integer k such that $\bigcup_{n=k}^{\infty}(A_{n+1}-A_n)=R'$ is contained in B. Then $N_2=B$ is the desired continuum of the second part of the conclusion, with $R'=R$. (4) No arc of R contains $R\cap B$, and also there is no integer k such that $\bigcup_{n=k}^{\infty}(A_{n+1}-A_n)$ is contained in B. In this case, $\bigcup A_n-B$ is the union of a countable number of disjoint open intervals, $I_1, I_2, I_3, \cdots$. Let $N_3=B\cup\bigcup I_n$. The upper-semicontinuous collection consisting of B and of the individual points of the intervals $\{I_n\}$ defines a map of N_3 onto a continuum of the third type of the conclusion of Theorem 1, satisfying the desired conditions.

From Theorem 1, we get an easy proof of Borsuk's theorem.

THEOREM 3. *If M is an arcwise connected, hereditarily unicoherent metric [or Hausdorff] curve, then M has the fixed-point property.*

PROOF. Note that a continuum of either of the first or third types described in Theorem 1 is not unicoherent, so that M contains neither of these. Next, *a hereditarily unicoherent arcwise connected continuum M contains no indecomposable subcontinuum.* For suppose that S is such an indecomposable subcontinuum of M. There is an arc A in M whose end points lie in different composants of M. Then A is not a subset of M, and $A \cup S$ is not unicoherent. Theorem 3 follows then from the next result, which seems to have escaped publication, and which shows that M can contain no continuum of the second type.

THEOREM 4. *If a hereditarily unicoherent continuum S contains a dense subset R that is the one-to-one continuous image of a half-open interval, but that contains no interior points, then S is indecomposable.*

PROOF. Suppose that S is the union of two proper subcontinua, A and B. Each has an interior, $\text{Int } A = S - B$ and $\text{Int } B = S - A$, relative to S. We may order the points of R by their order in the half-open interval, the image of the end point being the first point of R. Let a_1 be a point of $R \cap \text{Int } A$, b be a point of $R \cap \text{Int } B$ that follows a_1 in R, and a_2 be a point of $R \cap \text{Int } A$ that follows b in R. Then if a_1a_2 denotes the arc of R from a_1 to a_2, $A \cup a_1a_2$ is not unicoherent.

The join, in the sense of combinatorial topology, of a Cantor set and a point contains a subset R that is the continuous image of a half-open interval, that is dense in the join, and that has no interior, showing that the one-to-one property is required.

Theorem 1 does not imply Theorem 2. In fact, *for each integer $n > 1$, there is an arcwise connected, contractible and metric continuum containing no subcontinuum of any of the three types of Theorem* 1. Let X be a continuum of dimension $n-1$ that contains no arc; for example, the product of $n-1$ pseudo-arcs [**3**]. Let M be the join of X and a point p. If S is a continuum in M of one of the three types, $S-p$ cannot lie in one interval of the join, and the projection of $M-p$ onto X will map some arc of $S-p$ onto a nondegenerate continuum in X. However, Borsuk's hypothesis cannot hold in X, since a continuum of dimension greater than one cannot be hereditarily unicoherent. We can modify the example slightly, by replacing M by two such joins, having in common only one point, on the base of each, and show that *for each integer $n > 1$, there is an arcwise connected and noncontractible metric continuum containing no subcontinuum of any of the three types of Theorem* 1.

Kinoshita gave an example [2] of a contractible continuum that has no fixed point. Since it contains a 2-cell, it contains continua of all the types of Theorem 1. That result, however, does imply one fixed-point theorem for contractible continua.

THEOREM 5. *If M is a contractible Hausdorff continuum such that each two points are the end points of only one arc, then M has the fixed-point property.*

PROOF. Suppose that M contains a continuum N satisfying condition (1) of Theorem 1. The uniqueness of arcs shows that N cannot be a simple closed curve, so that one point-inverse under the mapping f of that condition is a nondegenerate continuum, B. The proof of part (1) of Theorem 1 shows that we can assume that N is the union of B and the continuous one-to-one image R of a half-open interval, $R \cap B$ consisting of the image of the end-point of that interval. Let $c: M \times I \to M$ be a contraction, satisfying $c(x, 1) = p$. By uniform continuity of c, for each positive number ϵ, there is a positive number δ such that if $d(x, y) < \delta$, then for all t in I, $d[c(x, t), c(y, t)] < \epsilon$.

Let y be a point of B not in R and not p. The set $c(y \times I) \cap R$ may be empty, but if not, it is connected. For if $c(y \times I) \cap R = H \cup K$, separated, then there exists an arc A_1 in R from a point h in H to a point k in K and there is an arc A_2 in $c(y \times I)$ from h to K, and $A_1 \cup A_2$ contains a simple closed curve. If e denotes the end point of R, which is in B, it is conceivable that e is not in $c(y \times I)$. It is not possible, however, that for some point x in R, $c(y \times I)$ contains the set R_x consisting of all the points z in R such that x is on the arc ez of R. For suppose that this occurred. Then $c(y \times I)$ contains B. Let U be a relatively open connected subset of the Peano continuum $c(y \times I)$ that contains e (which is in N), but does not contain x. Let x' be a point of $R_x \cap U$. There is an arc $x'e$ in U, and in R there are arcs ex, xx'. The union $x'e \cup ex \cup xx'$ contains a simple closed curve, which is impossible. We can thus conclude that $R - c(y \times I)$ contains a set $R_x - x$, for some x in R; x will be in $c(y \times I)$. If z is a point of R_x, then $c(z \times I)$ contains the arc xz of R; otherwise $xz \cup c(z \times I) \cup c(y \times I)$ contains a simple closed curve.

Now let $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ be a sequence of positive numbers approaching 0. For each ϵ_n, let δ_n be the corresponding number δ defined in the last sentence of the first paragraph of this proof, and let x_n be a point of R_x within δ_n of y. Then $d[c(x_n, t), c(y_0, t)] < \epsilon_n$ for all t in I. Let z be a fixed point of R_x; there is no loss in supposing that z is in each arc xx_n in R. Then by our last paragraph, z is in each set $c(x_n \times I)$. For each ϵ_n, then, $d(z, c(y \times I)) < \epsilon_n$, so that z belongs to the set $c(y \times I)$. But this is a contradiction.

Modifications of this argument dispose of each of the other two cases.

Either from Theorem 5 or, quicker, from Borsuk's theorem, it follows that a one-dimensional contractible continuum C has the fixed-point property, since every subcontinuum is homologically acyclic, so that C contains no simple closed curve.

Let C_1 be a continuum in the lower half xy-plane joining the point $(2, 0, 0)$ to the interval $[-3, -1]$ of the x-axis, C_1 being homeomorphic to the closure of the graph of $y=\sin 1/x$, $0<x\leqq\pi$, with the interval $[-3, -1]$ corresponding to the limiting interval of the graph. Let C_2 be the image of C_1 under the rotation of the xy-plane about the origin through an angle of π. Let L_1 and L_2 be straight-line intervals joining $(2, 0, 0)$ and $(-2, 0, 0)$ to $(0, 0, 1)$. Let R be a set homeomorphic to a half-open interval that (1) has only $(0, 0, 1)$ in common with $C_1 \cup C_2 \cup L_1 \cup L_2$ and (2) "spirals down" to $C_1 \cup C_2$ in such a way that (a) there is a sequence of arcs $X_1, X_2, X_3, \cdots$ filling up R such that $X_i \cap X_j$ is empty for $j \neq i+1, i-1$, and is an end point of each for $j=i+1, i-1$, and (b) $C_1=\lim X_{2j}$ and $C_2=\lim X_{2j+1}$. Let $M=C_1 \cup C_2 \cup L_1 \cup L_2 \cup R$. Then M is arcwise connected by unique arcs, and is compact. We define a continuous map $f\colon M \rightarrow M$ that has no fixed point. Let $f_1\colon M \rightarrow M$ be a map that on $C_1 \cup C_2 \cup L_1 \cup L_2$ is the rotation of E^3 about the Z-axis through an angle of π, and that on R is the identity; f_1 is not continuous. Let $f_2\colon M \rightarrow M$ be a map that is a homeomorphism on R and maps each arc X_n onto X_{n+1}; that is the identity on $C_1 \cup C_2$, and that maps each set L_j, $j=1, 2$, homeomorphically onto $L_j \cup X_1$, the points $(2, 0, 0)$ and $(-2, 0, 0)$ being kept fixed; f_2 is not continuous either. The composition $f=f_2f_1$, however, is continuous, and no point is left fixed.

I have no such example in the plane, nor do I have a continuum M that does not have the fixed-point property for homeomorphisms.

Bibliography

1. K. Borsuk, *A theorem on fixed points*, Bull. Acad. Polon. Sci. Cl. III vol. 2 (1954) pp. 17–20.

2. Shin'ichi Kinoshita, *On some contractible continua without fixed-point property*, Fund. Math. vol. 40 (1953) pp. 96–98.

3. E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 581–594.

4. L. E. Ward, Jr., *A fixed point theorem for chained spaces*, Pacific J. Math. vol. 9 (1959) pp. 1273–1278.

5. G. S. Young, *The introduction of local connectivity by change of topology*, Amer. J. Math. vol. 68 (1946) pp. 479–494.

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