DECOMPOSING 3-SPACE INTO CIRCLES AND POINTS

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1. **Introduction.** R. H. Bing and M. L. Curtis have shown that if $\mathcal{G}$ is an upper semicontinuous decomposition of E^n whose only nondegenerate elements are two compact continua, then the decomposition space G associated with $\mathcal{G}$ can be imbedded in E^{n+1}, [1]. Accordingly a decomposition of E^3 into two disjoint circles and points not on the circles can be imbedded in E^4. In the same paper Bing and Curtis exhibited a decomposition of E^3 into twelve mutually disjoint circles and points not on the circles so that the decomposition space could not be imbedded in E^4. The method was to show that the space contained a 2-dimensional polyhedron, the Menger polyhedron M_2, which Flores in [2] had shown could not be imbedded in E^4. It was conjectured in [1] that if the only nondegenerate elements were three circles linked by pairs, then the decomposition space could not be homeomorphically mapped into E^4. R. P. Goblirsch proved, however, that this conjecture was false (see [3]). Below, by modifying the construction of Bing and Curtis and slightly strengthening Flores' result, we give an example of a decomposition of E^3 into six circles and the points not on the circles so that the decomposition space cannot be imbedded in E^4.

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2. **Description of the Bing-Curtis example.** Consider the geometric 2-complex M_2, the Menger 2-polyhedron, consisting of three triples of vertices a_i, b_i, c_i, $i=1, 2, 3$, and all triangles of the form (a_i, b_j, c_k). In Figure 1 is depicted a construction by which Bing and Curtis imbed in E^3 the subpolyhedron of M_2 consisting of all triangles of M_2 except those of the form (a_i, b_i, c_i), $i=1, 2, 3$.

We will describe the construction in stages. Take a circular disk in the (x, y) plane and mark off in order at equal distances on its perimeter the points a_1, b_2, a_3, b_1, a_2, and b_3. Denote by c_2 the center

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of the disk. Taking radii from c_2 to each of the six designated points divides the disk into six sectors (a_i, b_j, c_2), $i \neq j$, which correspond to triangles in M_2.

Choose two points on the perpendicular to the (x, y) plane through c_2 which lie at a distance from c_2 equal to the radius of the disk, c_1 above c_2 and c_3 below c_2. Now take the spherical double cone over the

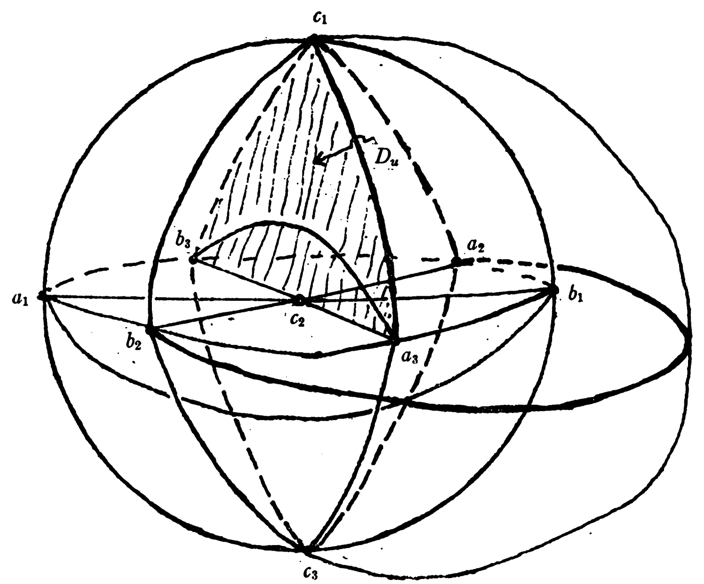

FIGURE 1

boundary of the disk with vertices c_1 and c_3. The upper hemisphere is divided into six spherical triangles (a_i, b_j, c_1), $i \neq j$; the lower hemisphere, into "triangles" (a_i, b_j, c_3) $i \neq j$.

The curvilinear complex so far constructed in E^3 has three complementary domains, an unbounded domain W and two bounded domains, U above the (x, y) plane, V, below.

Let D_U be the open disk which is the intersection of U with the plane through the 1-simplices (b_3c_2), (c_2a_3), (a_3c_1), and (c_1b_3). In $D_U \cup a_3 \cup b_3$ take a circular arc with endpoints a_3 and b_3. Similarly utilizing a planar open disk D_V in V, we fit the "triangles" (a_3, b_3, c_2), (a_3, b_3, c_1), (a_1, b_1, c_2), and (a_1, b_1, c_3) into the polyhedron.

In the (x, y) plane take a circular arc in W with endpoints a_2 and b_2. By taking curvilinear cones over (a_2b_2) from c_1 and c_3, we fit the "triangles" (a_2, b_2, c_1) and (a_2, b_2, c_3) into the complex.

We now cut out the interiors of nine disks and add three annular rings to this complex, as follows: In (a_3, b_3, c_1) cut out the interior of a disk D_1 so that the boundary curve J_1 intersects the boundary of (a_3, b_3, c_1) precisely in c_1. In the triangles (a_1, b_2, c_2) and (a_3, b_1, c_2) cut out disks D_2 and D_3 whose boundaries J_2 and J_3 intersect the boundaries of their respective triangles in a_1 and b_1. Using the holes cut out of the figure, an annular ring A_1 is placed with one boundary curve along the 1-chain $(a_1b_1)(b_1c_1)(c_1a_1)$ so that A_1 intersects the figure only along one of its boundary curves. Let the free boundary curve be called K_1. See Figure 2.

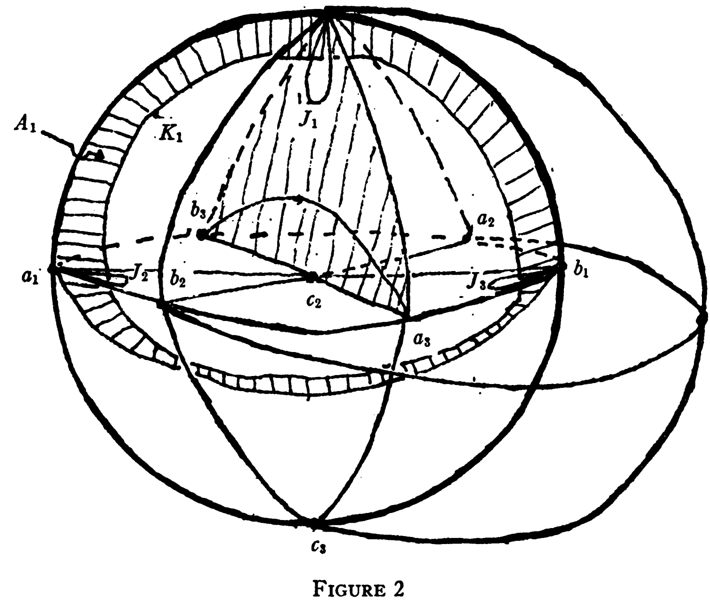

FIGURE 2

In a similar manner by cutting out disks E_1, E_2, E_3 with boundaries J_1', J_2', J_3' intersecting the 1-skeleton of the figure only in a_2, b_2, c_2 respectively, we add an annular ring A_2 with one boundary curve along $(a_2b_2)(b_2c_2)(c_2a_2)$ and with a free boundary curve K_2. An annulus A_3 is fitted along the edges $(a_3b_3)(b_3c_3)(c_3a_3)$ running through cut-out disks F_1, F_2, F_3, with boundaries J_1'', J_2'', J_3''. A_3 intersects the previous figure only along the aforementioned edge path, and we call its free boundary curve K_3. We call P the polyhedron so obtained.

We note that K_1, K_2, K_3 are linked by pairs, J_i links K_1 but none of the other added curves, J_i' links K_2, and J_i'' links K_3. The twelve simple closed curves K_i, J_i, J_i', J_i'' $(i=1, 2, 3)$ are all tame, and there is a homeomorphism of E^3 onto itself taking the twelve curves onto a collection of twelve circles linked, of course, in a similar manner. This implies that the decomposition space described by using these twelve curves as nondegenerate elements is homeomorphic to one in which the nondegenerate elements are circles.

The Bing-Curtis example is the decomposition of E^3 into the twelve curves and the remaining points. The decomposition space contains a homeomorph of M_2, namely the image of P. Identifying the curves J_i, J_i', J_i'' to points closes up the holes made by cutting out D_i, E_i, F_i; identifying the curves K_i to points adds in the triangles (a_i, b_i, c_i), $i=1, 2, 3$, in the decomposition space.

We can improve the example to one with only nine nondegenerate elements whose decomposition space still contains M_2. We eliminate J_1, J_1', J_1'' by the device of using "pinched annuli" for the A_i, that is sets homeomorphic to a disk with two points on the boundary identified; the idea is that such a figure still has two boundary curves such that if either is identified to a point, the identification image is again a disk. For example throwing away J_1, we leave (a_3, b_3, c_1) untouched and pass the pinched annulus A_1 with one boundary curve on $(a_1b_1)(b_1c_1)(c_1a_1)$ through the holes at D_2 and D_3. The other simple closed curve K_1 on the boundary of A_1 is free except at the point c_1. Identifying K_1, J_2, J_3 to points gives back all the triangles as before. The decomposition space with the nine curves K_1, K_2, K_3, J_i, J_i', J_i'' $(i=2, 3)$ as nondegenerate elements is the example mentioned.

3. **Our example using six curves.** It seems impossible to construct a decomposition space containing M_2 using fewer than nine circles. To construct the example promised in the Introduction, we modify the original Bing-Curtis twelve curve example as follows. We cut out only the disks D_1, E_1 and F_3 bounded by the curves J_1, J_1' and J_3'', eliminating the curves J_i, J_i', J_{i-1}'', $i=2, 3$. We relabel J_1, J_1' and J_3'' simply as J, J' and J''. The annuli A_i described above with edges K_i are retained. We call Q the polyhedron so obtained. We consider now the decomposition of E^3 in which the six curves J, J', J'', K_i, $i=1, 2, 3$, are the nondegenerate elements. The image of Q in the decomposition space we denote by N.

N is a polytope consisting of the union of twenty-seven topological triangles labelled in the same way as the triangles of M_2. We shall describe a map f of M_2 onto N which has the following property: for each pair of disjoint closed 2-simplexes σ and τ, $f(\sigma)$ and $f(\tau)$ are also disjoint. f maps the boundary of a 2-simplex $(a_ib_jc_k)$ isomorphically onto the boundary of the corresponding triangle in N with vertices going into corresponding vertices. Note that f restricted to the

1-skeleton of M_2 is an isomorphism. The annulus A_1 intersects the triangle (a_1, b_2, c_2) in Q in a line segment r_1 which joins a point on K_1 (the free boundary of A_1) to the vertex a_1. Except for one endpoint r_1 lies in the interior of (a_1, b_2, c_2). Similarly A_1 intersects (a_3, b_1, c_2) in a segment r_2 which joins a point on K_1 to the vertex b_1; except for one endpoint, r_2 lies in the interior of (a_3, b_1, c_2). We extend f to map homeomorphically the interior of the simplex (a_1, b_1, c_1) in M_2 onto the interior of the image of A_1 in N in such a fashion that the barycenter β goes into the image of K_1 and the closed line segments $s_1 = a_1\beta$ and $s_2 = b_1\beta$ go onto the images of r_1 and r_2 in N, respectively. In a similar fashion f takes the interior of (a_1, b_2, c_2) in M_2 onto the image in N of the interior of the triangle (a_1, b_2, c_2) from Q; the segment in M_2 which connects the vertex a_1 with the barycenter of (a_1, b_2, c_2) is mapped onto the image of r_1 in N. Similarly f is extended to the interior of (a_3, b_1, c_2) in M_2, mapping the segment which joins the barycenter of this simplex to the vertex b_1 onto the image of r_2 in N.

Let us now describe the self-intersections caused by f on the images of the simplexes mentioned. $f(a_1, b_1, c_1)$ meets $f(a_1, b_2, c_2)$ exactly in the image of r_1; $f(a_1, b_1, c_1)$ meets $f(a_3, b_1, c_2)$ precisely in the image of r_2; $f(a_1, b_2, c_2)$ meets $f(a_3, b_1, c_2)$ only in the two points $f(c_2)$ and the point in N, corresponding to the curve K_1, which is also the image of the barycenters of the three 2-simplexes. Notice that the self-intersections are produced only in the images of the interiors of the simplexes and that in M_2 each pair of these simplexes has a vertex in common.

f can be extended analogously over the remaining simplexes of M_2 in such a way that the desired property is preserved. The theorem proved in the next section guarantees that N cannot be imbedded in E^4 since if we let the C, f, and D used in the statement of the theorem be M_2, f and N, the hypotheses of the Theorem are satisfied for $n=2$. Since our example contains N, it in turn cannot be imbedded in E^4.

We note that each pair of K_1, K_2, K_3 are linked and J, J', J'' link only K_1, K_2, K_3, respectively. The decomposition of E^3 using these six curves is equivalent to a decomposition of E^3 using six circles similarly linked. We will complete the proof that this six curve decomposition space cannot be imbedded in E^4 by proving in the next section that N, the singular image of M_2, cannot be imbedded in E^4.

4. A strengthening of Flores' result. In order to obtain a proof of the theorem required, we have found it necessary to reprove many of the results mentioned in Flores' paper [2], which is essentially an outline. For the sake of completeness we include our own proofs; the form of the argument, i.e. the statements of many of the lemmas, is due to Flores. Our final theorem strengthens Flores' main result and is proved in an analogous manner.

Consider $A \times B \times I$, the Cartesian product of two disjoint spaces A and B with the unit interval $[0, 1]$. The *join* of A and B, which is designated by $V(A, B)$, is an identification space derived from $A \times B \times I$ as follows. For each $a \in A$, identify $(a, B, 0)$ with a. For each $b \in B$, identify $(A, b, 1)$ with b. Thus A and B are imbedded in $V(A, B)$, which may be thought of as a continuous collection of arcs connecting each pair of points one of which is from A, the other from B. Let ϕ be the identification function sending $A \times B \times I$ onto $V(A, B)$. The latter is topologized to make ϕ continuous, i.e. U is open in $V(A, B)$ if $\phi^{-1}(U)$ is open.

We consider the join operation to be symmetric so that $V(A, B)$ and $V(B, A)$ are the same space. If B is a point p, then $V(A, p)$ is also called the (topological) cone over A with vertex p. If B is a 0-sphere, $V(A, B)$ is called the suspension of A or the double cone over A with vertices in B. It is easy to see that $V(A, p)$ is homeomorphic to $A \times I$ with $(A, 1)$ identified as a point. Similarly $S(A)$, the suspension of A, is equivalent to $A \times [-1, 1]$ with $(A, -1)$ and $(A, 1)$ identified to separate points. It is also quickly verified that $V(A, B) - (A \cup B)$ is homeomorphic to $A \times B \times (0, 1)$.

If two spaces X and Y are homeomorphic, we shall write $X \approx Y$. For $n \geqq 0$, S^n will always mean the boundary of the unit ball in E^{n+1}.

LEMMA 1. $V(S^p, S^q) \approx S^{p+q+1}$.

PROOF. This is, of course, well known, but later we shall need to use the particular homeomorphism described below.

Let us consider S^p and S^q imbedded in E^{p+q+2} in the following way,

$$S^p = \{x \mid x = (x_1, \cdots, x_{p+1}, 0, \cdots, 0), \|x\| = 1\},$$
$$S^q = \{y \mid y = (0, \cdots, 0, y_1, \cdots, y_{q+1}), \|y\| = 1\}.$$

If $(x, y, t) \in V(S^p, S^q)$, let $\phi(x, y, t) = (1-t)^{1/2}x + t^{1/2}y$. ϕ is a homeomorphism of $V(S^p, S^q)$ onto S^{p+q+1}.

By a complex C we will always mean a countable locally finite complex; $|C|$ will be the underlying space with the barycentric metric. We will suppose that there is a fixed ordering $v_1, v_2, v_3, \cdots$ on the set of vertices of C which will be designated as C_v. Simplices will be considered to be closed. It will be understood that if a subcomplex of C contains a simplex σ, it contains the faces of σ.

Suppose K and L are disjoint complexes. The *join* of K and L,

$K \circ L$, is a complex consisting of all simplices σ such that $\sigma_v \subseteq K_v \cup L_v$ and $\sigma_v \cap K_v$, $\sigma_v \cap L_v$ are vertices of a simplex in K, L, respectively.

The next lemma is, again, well known, but the particular homeomorphism described will be used later.

LEMMA 2. *Let K and L be disjoint complexes. Then $V(|K|, |L|) \approx |K \circ L|$.*

PROOF. Given any pair of simplices $\sigma \in K$, $\tau \in L$, we will define a homeomorphism f: $V(\sigma, \tau) \rightarrow |\sigma \circ \tau|$. By the local finiteness of K and L it may be checked that these mappings may be fitted together to give the required homeomorphism.

For each point $x \in V(\sigma, \tau)$, since points of σ, τ have barycentric coordinates in $|K|$, $|L|$, respectively, x is of the form $(\sum \lambda_i v_i^K, \sum \mu_j v_j^L, t)$ and $f(x) = (1-t) \sum \lambda_i v_i^K + t \sum \mu_j v_j^L \in |\sigma \circ \tau|$.

Lemma 2 allows us to identify $V(|K|, |L|)$ and $|K \circ L|$ by the homeomorphism f, which in turn enables us to use equivalent methods of designating the points of the join of $|K|$ and $|L|$; as we shall see shortly, it is convenient to retain both notations.

Let C be an n-dimensional complex and C' a disjoint copy of C under an isomorphism θ. Then $U(C)$ will be the subcomplex of $C \circ C'$ consisting of all $(2n+1)$-dimensional simplices $\sigma \circ \tau'$ where σ and $\tau (=\theta^{-1}(\tau'))$ are disjoint n-simplices in C. We also denote $|U(C)|$ by $U(C)$.

Associated with C is another space $L(C)$ which is a subset of $V(|C|, p) \times V(|C|, p)$. $L(C)$ consists of all pairs (α, β) for which one coordinate lies in a simplex σ, the other coordinate lies in some $V(\tau, p)$, and σ and τ are disjoint n-simplices in C. If $(\alpha, \beta) \in L(C)$, then $(\beta, \alpha) \in L(C)$, and the two points are said to be *symmetric* in $L(C)$.

LEMMA 3. *$L(C)$ and $U(C)$ are homeomorphic.*

PROOF. Flores defines a geometric mapping g of $L(C)$ onto $U(C)$ which is clearly one-to-one and onto. Using the simplicial definition of join we give an equivalent characterization of g in terms of barycentric formulae which exhibit the bicontinuity of g more clearly and which shall also be needed in the next step.

Let (α, β) be a point of $L(C)$ for which α and β both belong to $|C|$. Then g maps the line segment $(\alpha, V(\beta, p))$ affinely onto half the line $V(\alpha, \beta')$ connecting α and $\beta' (=\theta(\beta))$ in $U(C)$ so that $g((\alpha, p)) = \alpha$ and $g((\alpha, \beta))$ is the midpoint $(\alpha, \beta', 1/2)$. Similarly g maps $(V(\alpha, p), \beta)$ affinely onto the other half of the line $V(\alpha, \beta')$ so that $g((p, \beta)) = \beta'$.

We will now exhibit formulae for g. Suppose $\alpha = \sum \lambda_i v_i \in \sigma$ and

$\beta=\sum\mu_i v_i\in\tau$ where σ and τ are disjoint n-simplices of C. Then $x=(\alpha,(\beta,p,t))=(\sum\lambda_i v_i,(1-t)\sum\mu_i v_i+tp)$ and $g(x)=(\alpha,\beta',(1-t)/2)$ $=((1+t)/2)\sum\lambda_i v_i+((1-t)/2)\sum\mu_i\theta(v_i)$. Also $y=((\alpha, p, t), \beta)$ $=((1-t)\sum\lambda_i v_i+tp, \sum\mu_i v_i)$ and $g(y)=(\alpha, \beta', (1+t)/2)$ $=((1-t)/2)\sum\lambda_i v_i+((1+t)/2)\sum\mu_i\theta(v_i)$.

If $x, y\in U(C)$ and $g^{-1}(x)$, $g^{-1}(y)$ are symmetric in $L(C)$, we shall call x and y symmetric in $U(C)$. Using the formulae for g we shall find a direct meaning intrinsic to $U(C)$ for two points of $U(C)$ to be symmetric.

Suppose $x=\sum\lambda_i v_i+\sum\mu_i\theta(v_i)(\sum\lambda_i+\sum\mu_i=1)$ is a point of $U(C)$; let $\theta(x)=\sum\mu_i v_i+\sum\lambda_i\theta(v_i)$. θ is an involution (a period two homeomorphism) of $U(C)$ onto $U(C)$ which we shall call the *antipodal* map of $U(C)$; x and $\theta(x)$ will be called *antipodal* points of $U(C)$. It is easy to see that if $(1-t)x+ty=z\in U(C)$, $\theta(z)=(1-t)\theta(x)+t\theta(y)$.

LEMMA 4. *x and y are symmetric in $U(C)$ if, and only if, x and y are antipodal points.*

PROOF. Let $(\alpha,\beta)\in L(C)$, $\alpha=\sum\lambda_i v_i$, $\beta=\sum\mu_i v_i$. Consider the points $z=(\alpha,(\beta,p,t))$, $z'=((\beta,p,t),\alpha)$ of $L(C)$. Then $g(z)=((1+t)/2)\sum\lambda_i v_i$ $+((1-t)/2)\sum\mu_i\theta(v_i)$, $g(z')=((1-t)/2)\sum\mu_i v_i+((1+t)/2)\sum\lambda_i\theta(v_i)$, and $\theta(g(z))=g(z')$.

Now let x and y be antipodal in $U(C)$. If $x=\sum\lambda_i v_i$, $\theta(x)=y$ $=\sum\lambda_i\theta(v_i)$, then $g^{-1}(x)=(x,p)$, and $g^{-1}(y)=(p,x)$.

Suppose $x=\sum\lambda_i v_i+\sum\mu_i\theta(v_i)$, $\sum\lambda_i+\sum\mu_i=1$, $1/2\leqq\sum\lambda_i<1$. Then $y=\theta(x)=\sum\mu_i v_i+\sum\lambda_i\theta(v_i)$. Equivalently, since $\sum\mu_i\leqq 1/2$, $x=((1/\sum\lambda_i)\sum\lambda_i v_i, (1/\sum\mu_i)\sum\mu_i\theta(v_i), \sum\mu_i)$, $y=((1/\sum\mu_i)\sum\mu_i v_i$, $(1/\sum\lambda_i)\sum\lambda_i\theta(v_i), \sum\lambda_i)$; accordingly, $g^{-1}(x)=(\alpha,(\beta,p,1-2\sum\mu_i))$ where $\alpha=(1/\sum\lambda_i)\sum\lambda_i v_i$, $\beta=(1/\sum\mu_i)\sum\mu_i v_i$. Similarly since $1/2$ $\leqq\sum\lambda_i$, $g^{-1}(y)=((\beta, p, 2\sum\lambda_i-1), \alpha)$, and $1-2\sum\mu_i=2\sum\lambda_i-1$. Hence $g^{-1}(x)$ and $g^{-1}(y)$ are symmetric in $L(C)$. The case in which $1/2\leqq\sum\mu_i$ follows in the same way.

A space X will be called *absolutely knotted* (*absolut selbstverschlungen*, [2]) with respect to E^m if there is no mapping ϕ of $V(X,p)$ into E^m satisfying these conditions, (a) ϕ is one-to-one on X and (b) $\phi(X)$ $\cap\phi(V(X,p)-X)=0$.

Clearly no subset of E^{m-1} is absolutely knotted in E^m since $V(E^{m-1},p)$ may be imbedded in E^m. Thus if X is absolutely knotted with respect to E^m, $V(X,p)$ cannot be imbedded in E^m and X cannot be imbedded in E^{m-1}.

Define M_0 to be a set of three points, and $M_n=V(M_{n-1},M_0)$, $n>0$. M_n may be thought of as $n+1$ triples of points a_i, b_i, c_i, $i=1,\cdots,$ $n+1$ and all n-dimensional simplices having exactly one vertex in each triple.

LEMMA 5. *There is a homeomorphism of S^{2n+1} onto $L(M_n)$ under which antipodal points in S^{2n+1} correspond precisely to symmetric points in $L(M_n)$.*

PROOF. We shall actually exhibit a homeomorphism of S^{2n+1} onto $U(M_n)$ so that antipodal points in S^{2n+1} correspond to antipodal points in $U(M_n)$. The proof proceeds by induction on n.

Let a, b, c be the vertices of an equilateral triangle inscribed in the unit circle $S^1 \subseteq E^2$. Let $\theta(a), \theta(b), \theta(c) = a', b', c'$ be diametrically opposite to the corresponding points a, b, c. Then, as may be quickly established, $U(M_0)$ is homeomorphic to the hexagon inscribed in S^1 with vertices a, b', c, a', b, c'. The desired homeomorphism h_0 maps S^1 onto $U(M_0)$ by projection through the center of S^1.

Now suppose a homeomorphism h_{n-1} of S^{2n-1} onto $U(M_{n-1})$ has been found with the required properties. Flores gives the formula,

$$\begin{aligned} U(M_n) &\equiv U(V(M_{n-1}, M_0)) \equiv V(U(M_{n-1}), U(M_0)) \\ &\equiv V(S^{2n-1}, S^1) \equiv S^{2n+1}. \end{aligned}$$

$U(M_n)$ is the same as $U(V(M_{n-1}, M_0))$ by definition.

We may regard M_n as the geometric complex consisting of all n-dimensional simplices $\sigma \circ a$ so that σ is any $(n-1)$-dimensional simplex in M_{n-1} and $a \in M_0$. Then $U(M_n)$ may be regarded as the complex consisting of all $(2n+1)$-dimensional simplices $(\sigma \circ a) \circ (\theta(\tau) \circ \theta(b))$ where σ, τ are disjoint $(n-1)$-simplices in M_{n-1}, and a, b are distinct points of M_0. On the other hand $U(M_{n-1})$ consists of all $(2n-1)$-simplices $\sigma \circ \theta(\tau)$, $\sigma \cap \tau = 0$, $U(M_0)$, of all 1-simplices $a \circ \theta(b)$, $a \neq b$. Therefore $V(U(M_{n-1}), U(M_0))$ consists of all $(2n+1)$-simplices of the form $(\sigma \circ \theta(\tau)) \circ (a \circ \theta(b))$. From the simplicial definition of join there is an obvious homeomorphism of $U(V(M_{n-1}, M_0))$ onto $V(U(M_{n-1}), U(M_0))$ given by a vertex isomorphism. The antipodal map θ is well defined in both spaces and antipodal pairs in one correspond to antipodal pairs in the other by the vertex isomorphism.

Under the induction hypothesis and the case $n=0$, we have homeomorphisms h_{n-1}, h_0 of S^{2n-1}, S^1 onto $U(M_{n-1})$, $U(M_0)$, respectively. We consider S^{2n-1} and S^1 imbedded in E^{2n+2} exactly as in the proof of Lemma 1. Using the homeomorphism ϕ described there, we have $\phi: V(S^{2n-1}, S^1) \rightarrow S^{2n+1} \subseteq E^{2n+2}$. Now we define $h_n': V(S^{2n-1}, S^1) \rightarrow V(U(M_{n-1}), U(M_0))$ by $h_n'((x, y, t)) = (h_{n-1}(x), h_0(y), t) = (1-t)h_{n-1}(x) + th_0(y)$, and the image point may be considered to be in $U(M_n)$ by the identification mentioned above.

Let z and $-z$ be antipodal points of S^{2n+1}. There is a point (x, y, t) in $V(S^{2n-1}, S^1)$ so that $\phi((x, y, t)) = (1-t)^{1/2}x + t^{1/2}y = z$. It follows that

$-z=(1-t)^{1/2}(-x)+t^{1/2}(-y)=\phi((-x, \quad -y, \quad t))$. By hypothesis $\theta(h_{n-1}(x))=h_{n-1}(-x)$; since in addition $\theta(h_0(y))=h_0(-y)$,

$$\theta(h_n'\phi^{-1}(z)) = \theta((1-t)h_{n-1}(x)+th_0(y)) = (1-t)h_{n-1}(-x)+th_0(-y) = h_n'\phi^{-1}(-z).$$

The argument may clearly be reversed, so we may set $h_n=h_n'\phi^{-1}$.

THEOREM. *Suppose C is an n-dimensional complex so that there is a homeomorphism of S^{2n+1} onto $L(C)$ under which antipodal points of S^{2n+1} are carried onto symmetric points in $L(C)$. If f is a mapping of $|C|$ onto a metric space D so that for any pair σ, τ of disjoint n-simplices in C, $f(\sigma)\cap f(\tau)=0$, then D is absolutely knotted with respect to E^{2n+1}.*

PROOF. $L(C)\subseteq V(|C|, p)\times V(|C|, p)$. f induces a mapping F of $V(|C|, p)\times V(|C|, p)$ onto $V(D, p)\times V(D, p)$ defined by $F((x, p, t), (y, p, s))=((f(x), p, t), (f(y), p, s))$. Let us denote by $\mathfrak{L}$ the image of $L(C)$ under F.

Suppose $\alpha\in\sigma\in C$ and $\beta=(x, p, t)$, $x\in\tau\in C$, and σ, τ are disjoint n-simplices. Then $z=F(\alpha, \beta)$ and $z'=F(\beta, \alpha)$ are clearly symmetric in $\mathfrak{L}$. If we assume that $z=z'$, more explicitly we have $(f(\alpha), (f(x), p, t)) =((f(x), p, t), f(\alpha))$. This means $f(\alpha)=(f(x), p, t)$ which in turn implies $t=0$ or $f(\alpha)=f(x)\in f(\sigma)\cap f(\tau)$, a contradiction. Thus symmetric points in $L(C)$ are carried onto distinct symmetric points in $\mathfrak{L}$.

Now let ϕ be a mapping of $V(D, p)$ into E^{2n+1}. There is a map ψ of $\mathfrak{L}$ into E^{2n+1} which takes a point (γ, δ) of $\mathfrak{L}$ into the endpoint of W, the vector from $\phi(\gamma)$ to $\phi(\delta)$, translated to the origin so that its starting point is at the origin. $\psi(\gamma, \delta)$ and $\psi(\delta, \gamma)$ are symmetric with respect to the origin of E^{2n+1}.

We denote by h the homeomorphism of S^{2n+1} onto $L(C)$ assumed in the hypotheses. If x, $-x$ are antipodal in S^{2n+1}, $Fh(x)$ and $Fh(-x)$ are distinct symmetric points in $\mathfrak{L}$. By the Borsuk-Ulam Theorem there exists an antipodal pair x, $-x$ in S^{2n+1} so that $\psi Fh(x)=\psi Fh(-x)$; since the image points are also, as we have observed previously, symmetric about the origin, it follows that ψFh maps x and $-x$ into the origin.

Let $Fh(x)=(\gamma, \delta)$ so $Fh(-x)=(\delta, \gamma)$. Since $\psi(\gamma, \delta)$ is the origin, $\phi(\gamma)=\phi(\delta)$. If $\gamma\in D$, $\delta\in D$ implies ϕ is not one-to-one on D; if δ is a point of $V(D, p)-D$, $\phi(D)\cap\phi(V(D, p)-D)\neq 0$. Accordingly D is absoutely knotted with respect to E^{2n+1}.

5. **Remarks and questions.** Some questions may be raised in view of the preceding results.

QUESTION 1. Is there a decomposition of E^3 into n circles, $n<6$, and points not on the circles so that the decomposition space cannot be imbedded in E^4? The author conjectures that if $n=4$ the answer is no. The results of Bing-Curtis and Goblirsch definitely settle the question in the negative for $n<4$. An example of a collection of circles for which the question is open is one of five circles in E^3 so that each circle links exactly two others.

QUESTION 2. If each nondegenerate element is a polyhedral simple closed curve rather than a circle, what is the least number of nondegenerate elements so that the decomposition space cannot be imbedded in E^4? It would be useful here to know broad conditions under which a collection of tame simple closed curves is equivalent in E^3 (under a global homeomorphism) to a collection of circles.

QUESTION 3. In Question 2 allow one or more of the curves to be wildly imbedded in E^3.

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