ON THE KERNEL OF A TOPOLOGICAL SEMIGROUP WITH CUT POINTS

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W. M. Faucett [1] recently studied the structure of the kernel of a compact connected mob which has a point that cuts the kernel. L. W. Anderson [2] has characterized the cut point of a connected topological lattice. The main purpose of this paper is to find a lattice theoretic characterization of the kernel by means of cut points in the topological semigroups derived from topological lattices. Using the concept of B-covers [3], we shall define a suitable multiplication in topological lattices to illustrate the structure of the kernel by lattice diagrams. The fact that a special case of Faucett's Theorem (Theorem 2) can be obtained from Anderson's result (Lemma 5) is important.

1. Preliminaries. We recall that a topological lattice is a Hausdorff space, L, together with a pair of continuous functions $\wedge: L\times L \to L$ and $\vee: L\times L\to L$ which satisfy the usual conditions stipulated for a lattice. If X is a topological space and $p\in X$, we say that p is a cut point of X if $X\backslash p$ is not connected, i.e., if $X\backslash p = U\cup V$ such that $U\neq\Box\neq V$ and $U^*\cap V=\Box=U\cap V^*$, where by A^* we mean the closure of A.

Hereafter let S be a connected topological lattice which satisfies the modular law. Now we introduce a multiplication in S as follows:

(M) $xy=(a\vee x)\wedge(b\vee y)$ for two fixed elements a, b of S.

For any two elements a, b of a lattice S let

$$B(a, b) = \{x \mid (a \vee x) \wedge (b \vee x) = (a \wedge x) \vee (b \wedge x) = x\};$$

then $B(a, b)$ is called the *B-cover* of a and b [3]. We define a *mob* to be a Hausdorff space together with a continuous associative multiplication. Then S is a mob with respect to (M), for the multiplication is continuous since S is a topological lattice and moreover it is associative by Lemma 1.

2. **The kernel** $B(a, b)$ **of a mob** S.

LEMMA 1. *$x(yz)=(xy)z$ in S.*

PROOF. We have

Received by the editors December 12, 1959.

$$\begin{aligned} x(yz) &= (a \vee x) \wedge (b \vee ((a \vee y) \wedge (b \vee z))) \\ &= (a \vee x) \wedge (a \vee b \vee y) \wedge (b \vee z), \\ (xy)z &= (a \vee ((a \vee x) \wedge (b \vee y))) \wedge (b \vee z) \\ &= (a \vee x) \wedge (a \vee b \vee y) \wedge (b \vee z) \text{ by the modular law.} \end{aligned}$$

Lemma 2. *If* $x \in B(a, b)$, $y \in S$, *then* (i) $xx = x$, (ii) $xy \in B(a, b)$, $yx \in B(a, b)$.

Proof. (i) follows from the definition of $B(a, b)$.

(ii) $(a \vee xy) \wedge (b \vee xy)$

$$\begin{aligned} &= (a \vee ((a \vee x) \wedge (b \vee y))) \wedge (b \vee ((a \vee x) \wedge (b \vee y))) \\ &= (a \vee x) \wedge (a \vee b \vee y) \wedge (a \vee b \vee x) \wedge (b \vee y) \\ &= (a \vee x) \wedge (b \vee y) = xy \text{ by the modular law;} \end{aligned}$$

$(a \wedge xy) \vee (b \wedge xy)$

$$\begin{aligned} &= (a \wedge ((a \vee x) \wedge (b \vee y))) \vee (b \wedge ((a \vee x) \wedge (b \vee y))) \\ &= (a \wedge (b \vee y)) \vee (b \wedge (a \vee x)) \\ &= (a \vee (b \wedge (a \vee x))) \wedge (b \vee y) \\ &= (a \vee b) \wedge (a \vee x) \wedge (b \vee y) \text{ by the modular law.} \end{aligned}$$

Since $x \leqq a \vee b$ for $x \in B(a, b)$ we have $(a \vee b) \wedge (a \vee x) \wedge (b \vee y) = (a \vee x) \wedge (b \vee y) = xy$. Similarly we have $yx \in B(a, b)$.

Lemma 3. *Let* $p \in B(a, b)$; *then* Sp *is a minimal left ideal and* pS *is a minimal right ideal.*

Proof. We shall prove that $(xp)(Sp) = xp$ for $x \in S$, $p \in B(a, b)$. For $y \in S$ we have $(xp)(yp) = (xpy)p = ((a \vee x) \wedge (a \vee b \vee p) \wedge (b \vee y))p = (a \vee x) \wedge (a \vee b \vee p) \wedge (a \vee b \vee y) \wedge (b \vee p) = (a \vee x) \wedge (b \vee p) = xp$ since $p \leqq a \vee b$. Similarly we have $(pS)(px) = px$.

Lemma 4. *If* $p \in B(a, b)$, *then* $B(a, b) = SpS$.

Proof. By Lemma 2 we have $B(a, b) \supset SpS$. If we take $r \in pS \cap Sq$ for $q \in B(a, b)$, then we have $qr = q$ by Lemma 3, where $r = px$ for some $x \in S$. Accordingly we have $B(a, b) \subset SpS$.

As a consequence, we have the following theorem.

Theorem 1. $B(a, b)$ *is the kernel of a mob* S.

3. The structure of the kernel $B(a, b)$ with cut points.

Lemma 5 (L. W. Anderson). *If* S *is a connected topological lattice and if* $p \in S$ *then* p *is a cut point of* S *if, and only if,* $p \neq 0$, $p \neq I$ *and* $L = (p \vee L) \cup (p \wedge L)$.

The next theorem is a special case of Faucett's theorem [1, Theorem 1.3].

THEOREM 2. *Let S be a compact connected mob derived from a compact connected topological lattice introducing the multiplication* (M) *into it. If there exists a point $p\in S$ that cuts $B(a, b)$, then we have either*

(i) $B(a, b)=\{x \mid a\leqq x\leqq b\}=Sp$, *that is, $B(a, b)$ is a minimal left ideal, and every element of $B(a, b)$ is left zero for S*; *or*

(ii) $B(a, b)=\{x \mid b\leqq x\leqq a\}=pS$, *that is, $B(a, b)$ is a minimal right ideal, and every element of $B(a, b)$ is right zero for S.*

PROOF. Since p cuts $B(a, b)$, we have $B(a, b)=A\cup B$, where $A=\{x \mid x\leqq p\}$ and $B=\{x \mid x\geqq p\}$, by Lemma 5.

Now suppose that $a, b\leqq p$; then for any element $x\in B$ such that $x>p$, we have $(a\wedge x)\vee(b\wedge x)=a\vee b\leqq p<x$, that is, x does not belong to $B(a, b)$, a contradiction. Similarly the case where $a, b\geqq p$ does not occur. Thus we have either $a\leqq p\leqq b$ or $b\leqq p\leqq a$. In the first case, any element x such that either $x<a$ or $b<x$ does not belong to $B(a, b)$. Now we shall prove that $B(a, b)=\{x \mid a\leqq x\leqq b\}=Sp$. Let $p\in B(a, b)$, $x\in S$; then $xp=(a\vee x)\wedge(b\vee p)=(a\vee x)\wedge b=a\vee(b\wedge x)$ by the modular law. Then we have $a\leqq xp\leqq b$, hence $Sp\subset B(a, b)$.

Conversely, if we take $k\in B(a, b)$, then $kp=(a\vee k)\wedge(b\vee p)=k\wedge b=k$ since $a\leqq k$, $p\leqq b$. It follows that $B(a, b)\subset Sp$. Accordingly we have $Sp=B(a, b)$, and hence $B(a, b)$ is a minimal left ideal by Lemma 3.

Now let $x\in S$, $k\in B(a, b)$; then $kx=(a\vee k)\wedge(b\vee x)=k\wedge(b\vee x)=k$ since $k\leqq b$, that is, every element of $B(a, b)$ is a left zero for S. This completes the proof of (i). Similarly we can prove (ii).

4. **The case where no point cuts the kernel $B(a, b)$ for S.** Throughout this section we shall assume that there is no point that cuts the kernel $B(a, b)$ of the mob S derived from a topological lattice.

We can easily find that (i) if $a\leqq b$, then $B(a, b)$ is a minimal left ideal for S, (ii) if $b\leqq a$, then $B(a, b)$ is a minimal right ideal for S, (iii) if a, b are noncomparable, then $B(a, b)$ has the same structure as that in Lemma 4.

Let us define a two-sided ideal T of a mob S to be a *prime ideal* provided that whenever $S\backslash T$ is non-null then $S\backslash T$ is a submob. A submob in a mob S is a nonvoid set T contained in S such that $TT\subset T$. Now we shall find a necessary and sufficient condition for a two-sided ideal C containing $B(a, b)$ to be a prime ideal in the case where $S\backslash z=C\cup D$, $C\neq\square\neq D$ and $C^*\cap D=\square=C\cap D^*$. In this case we do not assume that S is connected.

LEMMA 6. *Let* $S \backslash B(a, b) \ni z$; *then* $zz \in B(a, b)$ *if, and only if,* $z \leqq a \vee b$.

PROOF. By the modular law, we have $(a \vee zz) \wedge (b \vee zz) = zz$, $(a \wedge zz) \vee (b \wedge zz) = (a \vee z) \wedge (b \vee z) \wedge (a \vee b)$. If $z \leqq a \vee b$, then we have $zz \in B(a, b)$. Conversely if $zz \in B(a, b)$, then $zz = (a \vee b) \wedge (a \vee z) \wedge (b \vee z) = (a \vee b) \wedge zz$, and hence $z \leqq (a \vee z) \wedge (b \vee z) = zz \leqq a \vee b$. Hence we have $z \leqq a \vee b$.

THEOREM 3. *Let* S *be a mob with respect to multiplication* (M), *and let* z *be an element of* S *such that* $S \backslash z = C \cup D$, $C \neq \square \neq D$, $C^* \cap D = \square = C \cap D^*$ *and* C *is a two-sided ideal containing* $B(a, b)$; *then* C *is a prime ideal if, and only if* $z > a \vee b$.

Proof. By Lemma 5, we have either (i) $y < z < x$ or (ii) $y > z > x$ for all $x \in C$, $y \in D$. If $z > a \vee b$, let $S \backslash C = \{z, D\} = T \ni y_1, y_2$; then $y_1, y_2 \geqq z > a \vee b$, and hence $y_1 y_2 = (a \vee y_1) \wedge (b \vee y_2) = y_1 \wedge y_2 \geqq z > a \vee b$, that is, $y_1 y_2 \in T$. Then C is a prime ideal. If $z \not> a \vee b$, then $z \leqq a \vee b$ by Lemma 5. It follows that $zz \in B(a, b) \subset C$ by Lemma 6, and then C is not a prime ideal. This completes the proof.

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