restricting the boundary points α_1, α_3 further than requiring them to belong to $[a, \alpha_2]$, $[\alpha_2, b]$, respectively.

Conditions are not imposed on $a_{11}(x)$, $a_{nn}(x)$. If these functions are identically zero over (α_1, α_3) Theorem 2 follows for weaker restrictions than (C), (D). For the $a_{1f}(x)$, $a_{ne}(x)$, $f=2, \cdots, n$, $e=1, \cdots, n-1$, it is sufficient to require that the positive functions be nonnegative, the negative functions be nonpositive and $a_{1n}(\alpha_2)$, $a_{n1}(\alpha_2)>0$.

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A MOORE SPACE ON WHICH EVERY REAL-VALUED CONTINUOUS FUNCTION IS CONSTANT

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F. B. Jones [2] recently gave an example of a Moore space Λ_∞ in which there exists a point x such that Λ_∞ is not completely regular at x. It is easy to modify the construction used by Jones so as to obtain a Moore space A in which there exist distinct points a and b such that for every real-valued continuous function f on A, $f(a)=f(b)$. Upon applying Urysohn's process of condensation of the singularities of the space A [4], in a manner similar to that used by Hewitt [1], there results a Moore space X on which every real-valued continuous function is constant.

Throughout this paper, J denotes the set of positive integers. A sequence is a function on J, and if f is a sequence and $n\in J$, then f_n denotes $f(n)$.

By a Moore space is meant a topological space X whose topology has a basis consisting of sets termed *regions*, satisfying the following condition (axiom 1_3, that is, parts 1, 2, and 3 of axiom 1, of [3]): There exists a sequence G such that (1) if $n\in J$, G_n is a collection of regions covering X, (2) if $n\in J$, $G_{n+1}\subset G_n$, and (3) if r is a region, $x\in r$, and $y\in r$, then there exists a positive integer n such that if $g\in G_n$ and $x\in g$, then $\bar{g}\subset(r-\{x\})\cup\{y\}$. The following characterization of a Moore space will be used in this paper: X is a Moore space if and only if X is a regular Hausdorff space for which there exists a sequence G of open coverings of X such that if U is an open set and

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$p \in U$, there exists a positive integer n such that if $g \in G_n$ and $p \in g$, then $g \subset U$ [5].

1. **Construction of the space** A. Jones, in his construction of the space Λ_∞, made use of a certain Moore space Λ, an infinite sequence $\Lambda_1, \Lambda_2, \Lambda_3, \cdots$ of disjoint spaces, each congruent with Λ, and an ideal point p [2]. Adjacent terms of the sequence are pieced together along their boundaries in a certain way. The space A is to be constructed by using a doubly infinite sequence $\cdots, \Lambda_{-2}, \Lambda_{-1}, \Lambda_0, \Lambda_1, \Lambda_2, \cdots$ of disjoint spaces, each congruent with Λ, and two ideal points, a and b. Adjacent terms of the sequence are pieced together along their boundaries as in the construction of Λ_∞. Neighborhoods of a are defined as those for p are in the case of Λ_∞, and neighborhoods of b are defined in an obvious manner. A is a Moore space of cardinal c, and a slight modification of Jones' proof that Λ_∞ is not completely regular at p shows that if f is a continuous real-valued function on A, then $f(a)=f(b)$.

2. **Construction of the space** X. Now there will be constructed a Moore space X on which every real-valued continuous function is constant. Consider a collection of c disjoint spaces, each homeomorphic with the space A. This collection may be well-ordered as an order of ordinal number Δ, the initial ordinal of the cardinal c; let $A^1, A^2, A^3, \cdots, A^\lambda, \cdots, \lambda<\Delta$, be one such order. If $\lambda<\Delta$, then A^λ has cardinal c and may be well-ordered as an order of ordinal number Δ; let $x_1^\lambda, x_2^\lambda, x_3^\lambda, \cdots, x_\beta^\lambda, \cdots, \beta<\Delta$, be one such order where x_1^λ is the point a of the space A^λ and x_2^λ is the point b of A^λ.

Let Q be the set of all ordered quadruples $(\lambda_1, \beta_1; \lambda_2, \beta_2)$ where each of λ_1, β_1, λ_2, and β_2 is an ordinal number less than Δ, and either $\lambda_1<\lambda_2$, or $\lambda_1=\lambda_2$ and $\beta_1<\beta_2$. It can be shown by transfinite induction that there exists a one-to-one function Γ with domain Q and range the set of ordinals less than Δ such that (1) $\Gamma(1, 1; 1, 2)=1$ and $\Gamma(\lambda_1, \beta_1; \lambda_2, \beta_2)>\lambda_2$ for all elements of Q other than $(1, 1; 1, 2)$. Let ϕ be a function such that

$$\phi\left[x_1^{\Gamma(\lambda_1,\beta_1;\lambda_2\beta_2)}\right] = x_{\beta_1}^{\lambda_1} \quad \text{and} \quad \phi\left[x_2^{\Gamma(\lambda_1,\beta_1;\lambda_2\beta_2)}\right] = x_{\beta_2}^{\lambda_2}.$$

The function ϕ maps x_1^λ, $1<\lambda<\Delta$, into A^γ for some γ less than λ, and similarly for x_2^λ.

For each ordinal λ and each ordinal β, $\lambda<\Delta$ and $2<\beta<\Delta$, x_β^λ is an *initial point*; x_1^1 and x_2^1 are also initial points.

Certain sequences of points of $\bigcup_{\gamma<\Delta} A^\gamma$ are defined to be *chains*. C is a chain if and only if C is a sequence, $x_{\beta_1}^{\lambda_1}, x_{\beta_2}^{\lambda_2}, x_{\beta_3}^{\lambda_3}, \cdots$, such that

$\lambda_1<\lambda_2<\lambda_3< \cdots <\Delta$, $x_{\beta_1}^{\lambda_1}$ is an initial point, and if $n\in J$, then $\phi(x_{\beta_{n+1}}^{\lambda_{n+1}})=x_{\beta_n}^{\lambda_n}$. Each chain contains one and only one initial point, and if x_β^λ is an initial point, then C_β^λ is the union of all chains with x_β^λ as an initial point. If x_β^λ and x_α^μ are two distinct initial points, then C_β^λ and C_α^μ are disjoint. Further, if i is either 1 or 2, and $1<\lambda<\Delta$, then there is only one set C_α^μ such that x_i^λ belongs to a chain of C_α^μ. If x_β^λ belongs to a chain of C_α^μ, then x_β^λ is a *co-ordinate* of C_α^μ. Let X be the collection of all sets C_β^λ for $\lambda<\Delta$ and $\beta<\Delta$.

Now a topology for X will be constructed so that the resulting space is a Moore space. Suppose that λ is an ordinal, $\lambda<\Delta$. The space A^λ is a Moore space. Let G^λ be a sequence of collections of regions of the space A^λ satisfying axiom 1_3 relative to A^λ. Suppose now that β is an ordinal, $\beta<\Delta$. Let g_β^λ be a sequence of regions such that (1) if $n\in J$, then $x_\lambda^\beta\in g_{\beta n}^\lambda$ and $g_{\beta n}^\lambda\in G_n^\lambda$, (2) if $n\in J$, then $(g_{\beta(n+1)}^\lambda)^-\subset g_{\beta n}^\lambda$, and (3) if $2<\beta$, then $x_1^\lambda\notin(g_{\beta 1}^\lambda)^-$ and $x_2^\lambda\notin(g_{\beta 2}^\lambda)^-$. The sequences g_1^λ and g_2^λ are to satisfy the additional condition that

$$(\overset{\lambda}{g}_{11})^- \cap (\overset{\lambda}{g}_{21})^- = \emptyset.$$

Regions for X will now be defined. Suppose that $x\in X$. There exist an ordinal λ, $\lambda<\Delta$, and an ordinal β, $\beta<\Delta$, such that $x=C_\beta^\lambda$. Suppose that $n\in J$. $W_{\beta n}^{\lambda 0}$ is the set of all C_α^λ which have initial points in $g_{\beta n}^\lambda$. $W_{\beta n}^{\lambda 1}$ is defined as follows: (a) If both $x_1^{\lambda+1}$ and $x_2^{\lambda+1}$ are co-ordinates of elements of $W_{\beta n}^{\lambda 0}$, then $W_{\beta n}^{\lambda 1}$ is the set of all $C_\alpha^{\lambda+1}$ with initial points in $(g_{1n}^{\lambda+1}\cup g_{2n}^{\lambda+1})$. (b) If $x_i^{\lambda+1}$ is a co-ordinate of an element of $W_{\beta n}^{\lambda 0}$ but $x_j^{\lambda+1}$ is not a co-ordinate of any element of $W_{\beta n}^{\lambda 0}$, $i, j=1$ or 2, $i\neq j$, then $W_{\beta n}^{\lambda 1}$ is the set of all $C_\alpha^{\lambda+1}$ with initial point in $g_{in}^{\lambda+1}$. (c) If neither $x_1^{\lambda+1}$ nor $x_2^{\lambda+1}$ is a co-ordinate of any element of $W_{\beta n}^{\lambda 0}$, then $W_{\beta n}^{\lambda 1}=\emptyset$.

Suppose that ν is an ordinal, $\lambda+\nu<\Delta$, and for each ordinal μ, $\mu<\nu$, $W_{\beta n}^{\lambda\mu}$ has been defined. Let $Y_{\beta n}^{\lambda\nu}=\bigcup_{\mu<\nu} W_{\beta n}^{\lambda\mu}$. Then $W_{\beta n}^{\lambda\nu}$ is defined as follows: (a) If both $x_1^{\lambda+\nu}$ and $x_2^{\lambda+\nu}$ are co-ordinates of elements of $Y_{\beta n}^{\lambda\nu}$, then $W_{\beta n}^{\lambda\nu}$ is the set of all $C_\alpha^{\lambda+\nu}$ with initial point in the set $(g_{1n}^{\lambda+\nu}\cup g_{2n}^{\lambda+\nu})$. (b) If $x_i^{\lambda+\nu}$ is a co-ordinate of an element of $Y_{\beta n}^{\lambda\nu}$ but $x_j^{\lambda+\nu}$ is not a co-ordinate of any element of $Y_{\beta n}^{\lambda\nu}$, $i, j=1$ or 2, $i\neq j$, then $W_{\beta n}^{\lambda\nu}$ is the set of all $C_\alpha^{\lambda+\nu}$ with initial point in $g_{in}^{\lambda+\nu}$. (c) If neither $x_1^{\lambda+\nu}$ nor $x_2^{\lambda+\nu}$ is a co-ordinate of any element of $Y_{\beta n}^{\lambda\nu}$, then $W_{\beta n}^{\lambda\nu}=\emptyset$.

Let $\mathcal{W}_{\beta n}^\lambda=\bigcup_{0\leq\mu<\Delta} W_{\beta n}^{\lambda\mu}$. R is a region in X if and only if for some positive integer n and some element C_β^λ of X, R is $\mathcal{W}_{\beta n}^\lambda$.

By making minor modifications in Hewitt's proof [1], one may show that X is a regular Hausdorff space.

Suppose that $n\in J$; let $\mathcal{K}_n$ be the set of all $\mathcal{W}_{\beta n}^\lambda$ for all C_β^λ belonging to X. It is clear that if $n\in J$, $\mathcal{K}_n$ is an open cover of X. It will now be shown that if $\mathcal{U}$ is a region in X and $p\in\mathcal{U}$, then there exists a positive integer n such that if $h \in \mathcal{H}_n$ and $p \in h$, then $h \subset \mathcal{U}$. From this it follows that X is a Moore space.

Suppose that $\mathcal{U}$ is a region; for some λ, β, and positive integer n, $\mathcal{U} = \mathcal{W}^\lambda_{\beta n}$. Suppose that $p \in \mathcal{U}$; for some μ and α, with $\lambda \leqq \mu$, $p = C^\mu_\alpha$. There are two cases:

(1) $\lambda = \mu$. If $\lambda > 1$, then $\alpha > 2$ and there exists a positive integer m such that if $j \in J$ and $j > m$, then $x^\mu_\alpha \notin (g^\lambda_{1j} \cup g^\lambda_{2j})$. If $\lambda = 1$, take m to be 1. Now suppose that $k \in J$ and $k > m$. Suppose further that $h \in \mathcal{H}_k$ and $C^\lambda_\alpha \in h$. Then for some γ, $h = \mathcal{W}^\lambda_{\gamma k}$. For clearly C^λ_α does not belong to any $\mathcal{W}^\sigma_{\epsilon k}$ for $\sigma > \lambda$. If there exists an ordinal σ, $\sigma < \lambda$, and an ordinal ϵ such that $C^\lambda_\alpha \in \mathcal{W}^\sigma_{\epsilon k}$, then one of x^λ_1 and x^λ_2 is a co-ordinate of an element of $\mathcal{W}^\sigma_{\epsilon k}$, and $x^\lambda_\alpha \in (g^\lambda_{1k} \cup g^\lambda_{2k})$. However, as $k > m$, $x^\lambda_\alpha \notin (g^\lambda_{1k} \cup g^\lambda_{2k})$.

Since $C^\mu_\alpha \in \mathcal{W}^\lambda_{\beta n}$, then $x^\mu_\alpha \in g^\lambda_{\beta n}$. As A^λ is a Moore space, there exists a positive integer q such that if $g \in G^\lambda_q$ and $x^\mu_\alpha \in g$, then $g \subset g^\lambda_{\beta n}$; clearly q may be taken so that $q > m$. Now suppose that $h \in \mathcal{H}_q$ and $C^\lambda_\alpha \in h$. Then as $q > m$, for some γ, $h = \mathcal{W}^\lambda_{\gamma q}$; since $x^\lambda_\alpha \in g^\lambda_{\gamma q}$, then $g^\lambda_{\gamma q} \subset g^\lambda_{\beta n}$ and hence $\mathcal{W}^\lambda_{\gamma q} \subset \mathcal{W}^\lambda_{\beta n}$. Thus if $h \in \mathcal{H}_n$ and $C^\lambda_\alpha \in h$, then $h \subset \mathcal{U}$.

(2) $\lambda < \mu$. In this case, $\alpha > 2$. Since $C^\mu_\alpha \in \mathcal{W}^\lambda_{\beta n}$, then one of x^μ_1 and x^μ_2 is a co-ordinate of an element of $\mathcal{W}^\lambda_{\beta n}$, and x^μ_α belongs to one of g^μ_{1n} and g^μ_{2n}. Suppose $x^\mu_\alpha \in g^\mu_{in}$, $i = 1$ or 2.

There exists a positive integer m such that if $j \in J$ and $j > m$, then $x^\mu_\alpha \notin (g^\mu_{1j} \cup g^\mu_{2j})$. Then, as in case (1), if $k \in J$, $k > m$, $h \in \mathcal{H}_k$, and $C^\mu_\alpha \in h$, then for some γ, $h = \mathcal{W}^\mu_{\gamma k}$. As A^μ is a Moore space, there exists a positive integer q such that if $g \in G^\mu_q$ and $x^\mu_\alpha \in g$, then $g \subset g^\mu_{in}$; clearly q may be taken so that $q > m$. Now suppose that $h \in \mathcal{H}_q$ and $C^\mu_\alpha \in h$. As $q > m$, for some γ, $h = \mathcal{W}^\mu_{\gamma q}$; since $x^\mu_\alpha \in g^\mu_{\gamma q}$, then $g^\mu_{\gamma q} \subset g^\mu_{in}$ and therefore $\mathcal{W}^\mu_{\gamma q} \subset \mathcal{W}^\lambda_{\beta n}$. Thus if $h \in \mathcal{H}_q$ and $C^\mu_\alpha \in h$, then $h \subset \mathcal{U}$.

The space X is a Moore space; that every real-valued continuous function on X is constant may be proved exactly as in Hewitt [1].

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