ON THE FIELD OF VALUES OF A MATRIX

LOUIS BRICKMAN

1. **Introduction.** The idea of the field of values of a complex $n\times n$ matrix C was introduced by O. Toeplitz [3]. It is the set of complex numbers defined by

$$(1.1)\qquad W(C) = \{V^T C\overline{V} \mid V^T\overline{V} = 1\}, \qquad (V \text{ an } n \times 1 \text{ vector}).$$

W is clearly a compact and connected set. Toeplitz showed in [3] that W has a convex outer boundary, and a short time later F. Hausdorff [1] proved that W itself is convex. Since then several investigations have been made concerning the geometry of W. An example of a recent one is the dissertation of R. Kippenhahn [2] in which W is described as the convex hull of a certain algebraic curve of degree n obtainable from C.

Writing

$$(1.2)\qquad C = A + iB,$$

the unique Hermitian decomposition of C, we can exhibit W in the real coordinate form

$$(1.3)\qquad W(A, B) = \{(V^T A\overline{V}, V^T B\overline{V}) \mid V^T\overline{V} = 1\}.$$

This representation suggests a generalization to more than two Hermitian forms, but it is easy to show by example that convexity does not survive this extension. It is known, however, (see [1; 3]) that for three forms the boundary of W is still convex.

We shall be concerned with the real analog of (1.3), i.e., the set

$$(1.4)\qquad R(A, B) = \{(V^T AV, V^T BV) \mid V^T V = 1\}; \qquad A, B, V \text{ real}.$$

With the understanding (henceforth in force) that A and B are real symmetric matrices, we obviously have

$$R(A, B) \subset W(A, B).$$

In §2 we shall show that for $n \geqq 3$ we actually have

$$(1.5)\qquad R(A, B) = W(A, B).$$

In other words the nonreal vectors in (1.3) only duplicate the contributions of the real ones. This phenomenon is familiar in the case of one Hermitian form based on a real matrix, but it does not occur in the general case of three or more forms. (See Remark 4.) Indeed,

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the proof of Theorem 2.2 shows that it will occur if and only if R is convex. Hence this convexity is the first result established.

In §3 the sets obtained by removing the conditions $V^T\overline{V}=1$ and $V^TV=1$ are described. Using the results of §2 we find that

$$\hat{W}(A, B) = \{(V^TA\overline{V}, V^TB\overline{V}) \mid V \text{ complex}\} \tag{1.6}$$

and

$$\hat{R}(A, B) = \{(V^TAV, V^TBV) \mid V \text{ real}\} \tag{1.7}$$

are convex cones with

$$\hat{W}(A, B) = \hat{R}(A, B) \tag{1.8}$$

for every n. Finally it is observed that the set

$$\hat{W}(H, K, L) = \{(V^TH\overline{V}, V^TK\overline{V}, V^TL\overline{V}) \mid V \text{ complex}\} \tag{1.9}$$

is a convex cone in three-space. (H, K, L, any Hermitian matrices.)

2. **Proof of** (1.5). We can easily establish, exactly as Toeplitz did for (1.1), that R has a convex outer boundary for all n. Let us rather recall the method of Hausdorff. He showed that the inverse image under (1.1) of every line

$$ax + by + c = 0$$

of the xy plane is a connected subset of the unitary unit sphere. From this it follows that every line intersects W in a connected set, whence W is convex. This method has to be modified for the real case (1.4), however, as the following example shows. Taking $n=3$ let

$$\begin{aligned} x &= V^TAV = v_1^2 - v_2^2 - v_3^2 \\ y &= V^TBV(\text{arbitrary}) \qquad (v_1^2 + v_2^2 + v_3^2 = 1). \end{aligned} \tag{2.1}$$

The inverse image under (2.1) of the line $x=1$ consists of the two points $(1, 0, 0)$ and $(-1, 0, 0)$, a disconnected set. In spite of this we can still conclude that $x=1$ intersects R in a connected set; viz., the above set consists of two connected components which are symmetric about $(0, 0, 0)$. Hence they both have the same (connected) image under (2.1), and the conclusion follows. This idea is the key to the following theorem.

THEOREM 2.1. *If $n \geqq 3$, $R(A, B)$ is convex.*

PROOF. It is convenient to work with $n-1$ dimensional projective space P_{n-1}. Thus R is the image of P_{n-1} under the mapping

$$x = V^TAV/V^TV,$$
$$y = V^TBV/V^TV. \tag{2.2}$$

We shall prove that the inverse image under (2.2) of every line

$$ax + by + c = 0$$

is connected in P_{n-1}. Such a set is defined by

$$aV^TAV/V^TV + bV^TBV/V^TV + c = 0$$

or

$$V^T(aA + bB + cI)V = 0. \tag{2.3}$$

Hence the theorem reduces to the assertion that a real projective hyperconic is a connected set. This can be shown by induction. If $n=3$, the smallest value being considered, (2.3) is an ordinary projective conic Q_2, which is connected. (The nondegenerate conics are all topological circles, and the degenerate cases of (2.3) are the null set, a point, a line, a pair of lines, all of P_{n-1}.) Suppose that all hyperconics Q_{n-1} in P_{n-1} are connected, $n \geqq 3$, and let Q_n be a hyperconic in P_n. Through two arbitrary points of Q_n construct a hyperplane P_{n-1}. Then $Q_n \cap P_{n-1}$ is a hyperconic Q_{n-1}, which is connected. (That $Q_n \cap P_{n-1}$ is a hyperconic follows at once from the fact that the equation of P_{n-1} can be taken to be $v_n=0$ without loss of generality.) Thus any two points of Q_n belong to a connected subset of Q_n. The theorem now follows.

REMARK 1. Although the unitary analog of Theorem 2.1 holds for any n, Theorem 2.1 fails for $n=2$. Let

$$x = v_1^2 - v_2^2,$$
$$y = 2v_1v_2.$$

Then R is the unit circle $x^2+y^2=1$. More generally, it can be shown that for $n=2$, R is either an ellipse, a circle, a line segment, or a point.

REMARK 2. The theorem also breaks down if the number of quadratic forms is increased. For example let $n=3$ and consider the three forms

$$x = v_1^2 - v_2^2,$$
$$y = 2v_1v_2,$$
$$z = v_3^2.$$

The intersection of R and the supporting plane $z=0$ is again the circle $x^2+y^2=1$. Hence R does not even have a convex boundary.

THEOREM 2.2. *If* $n\geqq 3$, $R(A, B)=W(A, B)$.

PROOF. We need only show that $W\subset R$. Let

$$(x, y) \in W(A, B).$$

Then for some complex unit vector V

$$x = V^T A\overline{V},$$
$$y = V^T B\overline{V}.$$

Let

$$V = V_1 + iV_2, \qquad (V_1, V_2 \text{ real}).$$

Then

$$x = (V_1^T + iV_2^T)A(V_1 - iV_2) = V_1^T AV_1 + V_2^T AV_2,$$

and similarly

$$y = V_1^T BV_1 + V_2^T BV_2.$$

Since $V_1^TV_1+V_2^TV_2=1$, it is clear that $(x, y)\in R$ if either $V_1=0$ or $V_2=0$. In all other cases we can write

$$x = V_1^TV_1(U_1^TAU_1) + V_2^TV_2(U_2^TAU_2),$$
$$y = V_1^TV_1(U_1^TBU_1) + V_2^TV_2(U_2^TBU_2),$$

where U_1 and U_2 are the unit vectors

$$U_1 = (V_1^TV_1)^{-1/2}V_1,$$
$$U_2 = (V_2^TV_2)^{-1/2}V_2.$$

Hence (x, y) is a convex combination of two points of R, and therefore belongs to R.

REMARK 3. Although we can have $R\neq W$ for $n=2$, it is now clear that W is the convex hull of R in any case.

REMARK 4. The generalization of Theorem 2.2 to more than two forms is false. The quadratic forms of Remark 2 and the corresponding Hermitian forms provide a simple counterexample.

The following result is a generalization of Theorem 2.2.

COROLLARY. *Let C be any complex $n\times n$ matrix, $n\geqq 3$, and let*

$$R(C) = \{V^T C V \mid V \text{ real}, V^T V = 1\}.$$

Then

$$R(C) = W\left(\frac{1}{2}C + \frac{1}{2}C^T\right).$$

PROOF. If

$$C = H + iK, \qquad (H, K \text{ Hermitian})$$

then

$$\frac{1}{2}C + \frac{1}{2}C^T = \operatorname{Re} H + i \operatorname{Re} K.$$

Hence

$$\begin{aligned} R(C) &= R(H, K) = R(\operatorname{Re} H, \operatorname{Re} K) = W(\operatorname{Re} H, \operatorname{Re} K) \\ &= W\left(\frac{1}{2}C + \frac{1}{2}C^T\right). \end{aligned}$$

3. **Arbitrary vectors.** We wish now to discuss the removal of the conditions $V^T\overline{V}=1$ and $V^TV=1$. For example in place of (1.1) we can define

$$\hat{W}(C) = \{V^T C \overline{V} \mid V \text{ complex}\}. \tag{3.1}$$

Then

$$\hat{W}(C) = \{rz \mid r \geqq 0, z \in W(C)\}.$$

This, together with the convexity of W, implies that $\hat{W}$ is a convex cone. Thus if r_1z_1 and r_2z_2 belong to $\hat{W}$, and if a_1 and a_2 are any non-negative numbers, then

$$\begin{aligned} a_1r_1z_1 + a_2r_2z_2 = (a_1r_1 + a_2r_2)(&a_1r_1(a_1r_1 + a_2r_2)^{-1}z_1 \\ &+ a_2r_2(a_1r_1 + a_2r_2)^{-1}z_2) \in \hat{W}(C). \end{aligned}$$

(This holds trivially if $a_1r_1+a_2r_2=0$.) Similar remarks apply to $\hat{W}(A, B)$ and, if $n\geqq 3$, to $\hat{R}(A, B)$, defined in (1.6) and (1.7) respectively. Hence Theorem 2.2 implies that $\hat{R}$ and $\hat{W}$ are equal convex cones. This conclusion holds even for $n=2$. To see this let

$$(0, 0) \neq (x, y) \in W(A, B), \qquad (rx, ry) \in \hat{W}(A, B).$$

Since W is compact, there is a positive number s such that (sx, sy)

belongs to the boundary of W. But the boundary of W is precisely R. (See Remarks 1 and 3.) Therefore

$$(sx, sy) \in R(A, B),$$

and

$$(rx, ry) = (r/s\, sx, r/s\, sy) \in \hat{R}(A, B).$$

Thus

$$\hat{W}(A, B) \subset \hat{R}(A, B),$$

and since the opposite containment is obviously correct, the above assertion is proved.

Finally, let H, K, L, be Hermitian matrices. Then

$$W(H, K, L) = \left\{ (V^T H \overline{V}, V^T K \overline{V}, V^T L \overline{V}) \mid V^T \overline{V} = 1 \right\}$$

has a convex boundary. By reasoning similar to that just used, we conclude that $\hat{W}(H, K, L)$, defined in (1.9), is a convex cone. Thus we have the following theorem.

THEOREM 3.1. *Let* $\hat{W}(A, B)$, $\hat{R}(A, B)$, $\hat{W}(C)$, *and* $\hat{W}(H, K, L)$ *be the sets defined in* (1.6), (1.7), (3.1), *and* (1.9) *respectively. Then each of these sets is a convex cone. Moreover,*

$$\hat{W}(A, B) = \hat{R}(A, B)$$

holds for every n.

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YALE UNIVERSITY