NOTE ON POLYHARMONIC FUNCTIONS[1]

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Let Δ denote the m-dimensional Laplace operator $\sum_{\mu=1}^{m} \partial^2/\partial x_\mu^2$ and let the positive number r be defined by $r^2 = \sum_{\mu=1}^{m} x_\mu^2$. If $u(P)$ is a harmonic function in the sphere $\overline{PO} = r < R$—i.e., $u(P)$ is continuous and has continuous first and second derivatives for $r < R$ and satisfies there the equation $\Delta u = 0$—and if, moreover, $u(P) > 0$ for $r < R$, then, according to a classical result (Harnack's inequality)

$$u(P) \leqq u(0) \frac{R^{m-2}(R+r)}{(R-r)^{m-1}} \cdot \tag{1}$$

It is the purpose of this note to show that there exists an inequality of the same general character as (1) if the hypothesis that $u(P)$ be harmonic is replaced by the considerably weaker assumption that $u(P)$ be a polyharmonic function of order n $(n > 1)$. The latter expression refers to a function which is continuous, has continuous derivatives up to the order $2n$, and is a solution of $\Delta^{(n)} u = 0$, where $\Delta^{(k)}$ is defined as $\Delta(\Delta^{(k-1)})$. Our precise result is the following:

If $u(P)$ is a non-negative polyharmonic function of order n in the m-dimensional sphere $\overline{OP} = r < R$, then

$$u(P) \leqq A R^{m-2} \frac{(R+r)}{(R-r)^{m-1}} \tag{2}$$

where

$$A = m!\, 2^{n+m-2} \sum_{\nu=0}^{n-1} c_\nu \left| \Delta^{(\nu)} u(0) \right| R^{2\nu} \tag{3}$$

and

$$c_\nu^{-1} = 2^\nu \nu!\, m(m+2) \cdots (m+2\nu-2). \tag{4}$$

The proof of (2) will be based on the fact [1] that a function $u(P)$ which is polyharmonic of order n in $r < R$ has a representation

$$u(P) = \sum_{\nu=0}^{n-1} u_\nu(P) r^{2\nu}, \qquad r < R, \tag{5}$$

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where the functions $u_\nu(P)$ are harmonic for $r<R$. In order to facilitate the writing we shall assume that $u(P)$ is regular in $r \leqq R$. No generality is lost by this assumption, since it can be made to hold by the preliminary coordinate transformation $r \to r(1-\epsilon)$, where ϵ is a small positive number, and ϵ can be made to tend to zero in the final result. If we regard the right-hand side of (5) as a polynomial in r^2 and set

$$v(P;\rho^2) = \sum_{\nu=0}^{n-1} u_\nu(P)\rho^{2\nu}, \tag{6}$$

it follows from the Lagrange interpolation formula that

$$u(P) = \sum_{\nu=1}^{n} v(P;R_\nu^2)\frac{F_\nu(r^2)}{F_\nu(R_\nu^2)}, \tag{7}$$

where

$$F_\nu(r^2) = \prod_{k=1;\,k\neq\nu}^{n} (R_\nu^2 - r^2), \tag{8}$$

and the R_ν are positive numbers which satisfy $r<R_1<R_2< \cdots <R_{n-1}<R_n=R$ and are otherwise arbitrary.

For a fixed value of ρ, the function $v(P;\rho^2)$ is harmonic for $r\leqq R$. If P is such that $\overline{OP}=\rho$, a comparison of (5) and (6) shows that $u(P)=v(P;\rho^2)$, and it follows from our hypotheses that $v(P;\rho^2)\geqq 0$. But $v(P;\rho^2)$ is a harmonic function of P, and the maximum principle shows therefore that $v(P;\rho^2)\geqq 0$ for all P such that $\overline{OP}\leqq\rho$. Since $R_\nu^2>r^2$, the expressions $v(P;R_\nu^2)$ in (7) will accordingly all be nonnegative. By (8), the value of $F_\nu(R_\nu^2)$ is positive if ν is odd, and negative if ν is even. We may thus conclude from (7) that

$$u(P) \leqq {\sum_{\nu=1}^{n}}' v(P;R_\nu^2)\frac{F_\nu(r^2)}{F_\nu(R_\nu^2)}, \tag{9}$$

where the prime indicates that the summation is to be extended only over the odd values of ν.

We now set

$$R_\nu^2 = r^2 + \frac{\nu}{n}(R^2 - r^2). \tag{10}$$

By (8), we then have $F_\nu(r^2)=\nu^{-1}n^{1-n}n!(R^2-r^2)^{n-1}$, and $F_\nu(R_\nu^2) =(\nu-1)!(n-\nu)!n^{1-n}(R^2-r^2)^{n-1}$. Inequality (9) will thus reduce to

$$u(P) \leqq {\sum_{\nu=1}^{n}}' \binom{n}{\nu} v(P;R_\nu^2),$$

where R_ν^2 is defined by (10). Since $v(P; R_\nu^2)$ is a non-negative harmonic function for $\overline{OP} \leqq R_\nu$, it follows from (1) and (6) that

$$v(P; R_\nu^2) \leqq \left(\sum_{k=0}^{n-1} u_k(0) R_\nu^{2k} \right) \frac{R_\nu^{m-2}(R_\nu + r)}{(R_\nu - r)^{m-1}} .$$

Setting

$$B = \sum_{k=0}^{n-1} | u_k(0) | R_\nu^{2k} \tag{11}$$

and observing that $\sum_{k=0}^{n-1} u_k(0) R_\nu^{2k} \leqq B$ and

$$\frac{R_\nu^{m-2}(R_\nu + r)}{(R_\nu - r)^{m-1}} < \frac{R^{m-2}(R + r)^m}{(R_\nu^2 - r^2)^{m-1}} = \left(\frac{n}{\nu}\right)^{m-1} \frac{R^{m-2}(R + r)}{(R - r)^{m-1}},$$

we obtain

$$u(P) \leqq B \frac{R^{m-2}(R + r)}{(R - r)^{m-1}} \sum_{\nu=1}^{n}{}' \left(\frac{n}{\nu}\right)^{m-1} \binom{n}{\nu}. \tag{12}$$

If k is a positive integer, we have $n/\nu < (k+1)(n+k)/(\nu+k)$, whence

$$\left(\frac{n}{\nu}\right)^{m-1} < m! \frac{(n + 1)(n + 2) \cdots (n + m - 1)}{(\nu + 1)(\nu + 2) \cdots (\nu + m - 1)}, \tag{13}$$

and therefore

$$\left(\frac{n}{\nu}\right)^{m-1} \binom{n}{\nu} < m! \binom{n + m - 1}{\nu + m - 1}.$$

Accordingly,

$$\sum_{\nu=1}^{n}{}' \left(\frac{n}{\nu}\right)^{m-1} \binom{n}{\nu} < m! \sum_{\mu=1}^{n+m-1}{}' \binom{n + m - 1}{\mu},$$

where the prime on the right-hand side indicates that the summation is to be extended only over indices μ such that $\mu \equiv m \pmod 2$. Since the value of this sum is 2^{n+m-2}, we thus conclude from (12) that

$$u(P) < 2^{n+m-2} m! \, B \frac{R^{m-2}(R + r)}{(R - r)^{m-1}} .$$

In view of (11), (5), and the fact that, for a harmonic function $w(P)$,

$$\Delta^{(k)}(r^{2k}w)_{P=0} = 2^k k! m(m + 2) \cdots (m + 2k - 2) w(0)$$

and $\Delta^{(k)}(r^{2\mu}w)_{P=0} = 0$ for $\mu \neq k$, this completes the proof of inequality (2).

For any special value of n, a more accurate inequality can be obtained by evaluating the sum on the right-hand side of (12) and avoiding the rather crude estimate (13). For instance, if $n=2$—i.e., in the case of a biharmonic function $u(P)$—the summation in (12) reduces to the term corresponding to $\nu=1$. A positive biharmonic function will therefore be subject to the inequality

$$u(P) \leqq 2^m[u(0) + 2m \mid \Delta u(0) \mid] \frac{R^{m-2}(R + r)}{(R - r)^{m-1}} \cdot$$

It may be pointed out that, as the example $u(P)=\overline{PQ}^2u_0(P)$, $\Delta u_0=0$, $u_0(P) \geqq 0$ shows, a non-negative polyharmonic function $u(P)$ may vanish at any point Q in the sphere $r<R$. Hence, there does not exist an analogue for polyharmonic functions of Harnack's "lower" inequality for harmonic functions.

As an application of the inequality (2) we prove the following result of Nicolesco [**2**; **3**].

If $u(P)$ is non-negative and polyharmonic of order n at all finite points P of the m-dimensional Euclidean space, then $u(P)$ reduces to a polynomial of order $2n-2$ in the Cartesian coordinates $x_1, \cdots, x_m$.

By assumption, (2) holds for all r and R such that $0 \leqq r<R<\infty$. Setting, in particular, $R=2r$, we obtain $u(P)<AC_1$, where C_1 is a constant. By (3), $A<C_2r^{2n-2}$, and thus

$$0 \leqq u(P) \leqq Cr^{2n-2}, \tag{14}$$

where C is again a constant. The functions $u_\nu(P)$ in (5) are harmonic for all finite non-negative values of r, and we may therefore expand them into series

$$u_\nu(P) = \sum_{k=0}^{\infty} r^k S_k^{(\nu)},$$

where the $S_k^{(\nu)}$ are spherical harmonics of order k and do not depend on r. Hence,

$$u(P) = \sum_{k=0}^{\infty} r^k \left(\sum_{\nu=0}^{n} r^{2\nu} S_k^{(\nu)} \right). \tag{15}$$

If S_k is any spherical harmonic of order k, it follows from the orthogonality properties that

$$\int_{U_m} u(P)S_k d\omega_m = r^k \sum_{\nu=0}^{n-1} A_k^{(\nu)} r^{2\nu},$$

where U_m is the unit sphere in m-space, $d\omega_m$ the area element on U_m, and

$$A_k^{(\nu)} = \int_{U_m} S_k^{(\nu)} S_k d\omega_m.$$

If A_k is a constant such that $|S_k| < A_k$, it follows from (14) that

$$\left| \int_{U_m} u(P) S_k d\omega_m \right| \leqq C A_k \Omega_m r^{2n-2},$$

where Ω_m is the area of U_m. Hence

$$\left| r^k \sum_{\nu=0}^{n-1} A_k^{(\nu)} r^{2\nu} \right| \leqq C_3 r^{2n-2}, \tag{16}$$

where C_3 is another constant. If $k > 2n-2$, this shows that

$$\lim_{r\to\infty} \left| \sum_{\nu=0}^{n-1} A_k^{(\nu)} r^{2\nu} \right| = 0,$$

and we must therefore have $A_k^{(\nu)} = 0$ for $\nu = 0, \cdots, n-1$ and $k > 2n-2$. Since S_k was an arbitrary spherical harmonic of order k, it follows that the functions $S_k^{(\nu)}$ vanish identically. The inequality (16) shows, moreover, that all $A_k^{(\nu)}$ for which $k+2\nu > 2n-2$ are necessarily zero. The corresponding spherical harmonics $S_k^{(\nu)}$ will therefore also vanish identically. Accordingly, the expansion (15) reduces to a polynomial in r of order not exceeding $2n-2$, and the result follows.

We finally show that the representation (7) provides a simple means of obtaining the Poisson formula analogue for polyharmonic functions without the explicit knowledge of the Green's function. We choose a small positive number ϵ such that $n\epsilon < R^2 - r^2$ and set $R_\nu^2 = R^2 - \epsilon\nu$. If $F_\nu(r^2)$ is the expression (8), we then have

$$(R_\nu^2 - r^2) F_\nu(r^2) = \prod_{k=1}^{n} (R^2 - r^2 - k\epsilon) \tag{17}$$

and

$$F_\nu(R_\nu^2) = (-1)^{n-\nu} \epsilon^{n-1} (\nu - 1)!(n-\nu)! \tag{18}$$

The function $v(P; R_\nu^2)$ is harmonic for $r \leqq R_\nu$ and coincides with $u(P)$ for $r = R_\nu (r = \overline{OP})$. It may therefore be represented by the Poisson integral

$$\frac{R_\nu^{m-2}(R^2 - r^2)}{\Omega_m} \int_{U_m} \frac{u(P_\nu) d\omega_m}{(R_\nu^2 - 2rR_\nu \cos\gamma + r^2)^{m/2}},$$

where Ω_m is the surface area of the m-dimensional unit sphere U_m,

$d\omega_m$ is the corresponding surface element, and γ is the angle between the vectors $\mathbf{OP}$ and $\mathbf{OP}_\nu$ ($\overline{OP}_\nu = R_\nu$). In view of (7), (17), and (18), we thus have

$$u(P) = \frac{\prod_{k=1}^{m}(R^2 - r^2 - k\epsilon)}{(n-1)!\epsilon^{n-1}\Omega_m}\int_{U_m}\left[\sum_{\nu=1}^{n}(-1)^{n-\nu}\binom{n-1}{\nu-1}\frac{R_\nu^{m-2}u(P_\nu)}{k(R_\nu)}\right]d\omega_m,$$

where $k(R_\nu) = (R_\nu^2 - 2rR_\nu \cos\gamma + r^2)^{m/2}$.

If $F(\rho^2)$ is a function having $n-1$ continuous derivatives at $\rho^2 = R^2$, we have

$$\begin{aligned}\phi &\equiv \sum_{\nu=1}^{n}(-1)^{n-\nu}\binom{n-1}{\nu-1}F(R_\nu^2)\\ &= \sum_{\nu=0}^{n-1}(-1)^{n-\nu-1}\binom{n-1}{\nu}F(R_{\nu+1}^2)\\ &= \sum_{\nu=0}^{n-1}(-1)^{\nu}\binom{n-1}{\nu}F(R_{n-\nu}^2)\\ &= \sum_{\nu=0}^{n-1}(-1)^{\nu}\binom{n-1}{\nu}F(R^2 - \nu\epsilon),\end{aligned}$$

and therefore

$$\lim_{\epsilon\to 0}\frac{\phi}{\epsilon^{n-1}} = \frac{\partial^{n-1}F(R^2)}{\partial(R^2)^{n-1}}\,.$$

Accordingly, a function $u(P)$ which is polyharmonic of order n in the m-dimensional sphere $r<R$, and has continuous partial derivatives up to the order $n-1$ in $r \leqq R$, has the representation

$$u(P) = \frac{(R^2 - r^2)^n}{(n-1)!\Omega_m}\int_{U_m}\frac{\partial^{n-1}}{\partial(R^2)^{n-1}}\left[\frac{R^{m-2}u(P')}{(R^2 - 2rR\cos\gamma + R^2)^{m/2}}\right]d\omega_m.$$

For biharmonic functions ($n=2$) an equivalent formula was found by Lauricella and Volterra (quoted in [3]).

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