ON THE SPACE OF SUBSETS OF A UNIFORM SPACE

A. P. ROBERTSON AND WENDY ROBERTSON[1]

The set $\mathcal{E}$ of closed nonempty subsets of a metric space E can be made into a metric space by means of the well-known Hausdorff metric (Hausdorff [3, §28]). When E is complete, so is $\mathcal{E}$ (see e.g. Kuratowski [6, §29, no. IV]; Price [7]; Bourbaki [1, Chapter IX, §2, ex. 7]). In a similar way, if E is a uniform space, the uniform structure on E induces on $\mathcal{E}$ a separated uniform structure, which we shall call the Hausdorff uniform structure on $\mathcal{E}$, following J. L. Kelley [5]. In this case, it is no longer true that the completeness of E implies that of $\mathcal{E}$ (although, if E is compact, $\mathcal{E}$ is then complete; see e.g. Bourbaki [1, Chapter II, §4, ex. 6]). An example of a complete uniform space E for which $\mathcal{E}$ is not complete is provided by taking for E any complete but not fully complete locally convex topological vector space, for the completeness of $\mathcal{E}$ implies the full completeness of E (see [5]). The first result of this paper shows that the subset of $\mathcal{E}$ consisting of all the (nonempty) compact sets in E is always complete, provided that E is complete. The second theorem gives an extension of this result, which, applied to a complete locally convex topological vector space, shows that the weakly compact sets also form a complete space. The third theorem is concerned with function spaces. The corresponding result for a space of linear mappings can be regarded as a generalization of the fact, that, for Banach spaces, the limit of a convergent sequence of weakly compact linear mappings is weakly compact. This is because convergence in norm of a sequence of linear mappings is precisely convergence, in the Hausdorff uniform structure on the range space, of the sequence of closed images of the unit ball.

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Let E be a separated uniform space and $\mathfrak{U}$ a base for the uniform structure on E. If $U \in \mathfrak{U}$ and $A \subseteq E$, we write $U(A)$ for the set of points y with $(x, y) \in U$ for some $x \in A$. On $\mathcal{E}$, the set of closed nonempty subsets of E, the sets

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$$W_U = \{(A, B): A \subseteq U(B) \text{ and } B \subseteq U(A)\}, \qquad U \in \mathfrak{U},$$

form a base for a separated uniform structure, called the *Hausdorff uniform structure.*

We collect some general facts about this structure. The net (B_α) in $\mathcal{E}$ is Cauchy if and only if to each $U \in \mathfrak{U}$ corresponds $\alpha(U)$ with $B_\alpha \subseteq U(B_\beta)$ for all $\alpha, \beta \geqq \alpha(U)$; it converges to B if and only if there exists $\alpha(U)$ with $\mathrm{N} \subseteq U(B_\alpha)$ and $B_\alpha \subseteq U(B)$ for all $\alpha \geqq \alpha(U)$. If t is a uniformly continuous mapping of E into another separated uniform space F, the mapping $A \to \overline{t(A)}$ of $\mathcal{E}$ into the set $\mathfrak{F}$ of closed nonempty subsets of F is uniformly continuous for the Hausdorff uniform structures on $\mathcal{E}$ and $\mathfrak{F}$. Hence if (B_α) is Cauchy, so is $(\overline{t(B_\alpha)})$ and if $B_\alpha \to B$ then $\overline{t(B_\alpha)} \to \overline{t(B)}$. If F is a closed subspace of E then $\mathfrak{F}$ is a closed subspace of $\mathcal{E}$. (For if $B \in \overline{\mathfrak{F}}$ and $U \in \mathfrak{U}$, then $B \subseteq U(A)$ for some $A \in \mathfrak{F}$; hence $B \subseteq U(F)$ for all $U \in \mathfrak{U}$ and so $B \subseteq \overline{F} = F$.) Any increasing net (B_α) of sets whose union B is precompact converges to $\overline{B}$. Any decreasing net which converges must converge to the intersection; so also does any Cauchy decreasing net of compact sets. If $B_\alpha \to B$ and each B_α is precompact, then so is B. Thus, in a complete space, if B_α converges to the (closed) set B and each B_α is compact, then so is B.

THEOREM 1. *Let E be a complete separated uniform space and $\mathcal{E}$ the space of closed nonempty subsets of E under the Hausdorff uniform structure. Then the nonempty compact subsets of E form a complete subspace of $\mathcal{E}$.*

PROOF. Since E is a separated uniform space, it can be embedded in a product $F = \times\{E_\gamma : \gamma \in \Gamma\}$ of complete metric spaces (see e.g. Kelley [**4**, Chapter 6, §16]). Let (B_α) be a Cauchy net of compact subsets of E. For each finite subset ϕ of Γ let p_ϕ be the natural projection mapping of F onto $F_\phi = \times\{E_\gamma : \gamma \in \phi\}$. Since p_ϕ is uniformly continuous, $(p_\phi(B_\alpha))$ is a Cauchy net of compact subsets of the complete metric space F_ϕ. Hence it converges, to the compact set A_ϕ, say. In the special case when $\phi = \{\gamma\}$, we shall write simply A_γ. Put $A = \times\{A_\gamma : \gamma \in \Gamma\}$ and, for each ϕ, $C_\phi = A \cap p_\phi^{-1}(A_\phi)$. Then each C_ϕ is compact, being a closed subset of a product of compact sets; we prove that C_ϕ decreases as ϕ increases. For let $\psi \subseteq \phi$ and denote by q the natural projection of F_ϕ onto F_ψ. Then clearly $q \circ p_\phi = p_\psi$, so that $q(p_\phi(B_\alpha)) = p_\psi(B_\alpha)$ for all α, and therefore $q(A_\phi) = A_\psi$. Thus

$$p_\psi(\overset{-1}{p_\phi}(A_\phi)) = (q \circ p_\phi \circ \overset{-1}{p_\phi})(A_\phi) = q(A_\phi) = A_\psi \text{ and so } C_\phi \subseteq C_\psi.$$

In particular $C_\phi \subseteq C_\gamma$ for each $\gamma \in \phi$, so that $C_\phi = \times\{A_\gamma : \gamma \notin \phi\} \times A_\phi$, which proves that each C_ϕ is nonempty. Since the sets C_ϕ are compact and decrease with increasing ϕ, their intersection C is nonempty and compact.

Next, $p_\psi(C)=A_\psi$. For if $\psi\subseteq\phi$, $p_\psi(C_\phi)=q(A_\phi)=A_\psi$; hence, if $x\in A_\psi$, the sets $C_\phi\cap p_\psi^{-1}(x)$, with $\psi\subseteq\phi$, are nonempty. They are also compact and decrease as ϕ increases, and so their intersection $C\cap p_\psi^{-1}(x)$ is nonempty. Thus $A_\psi\subseteq p_\psi(C)$. But $p_\psi(C)\subseteq p_\psi(C_\psi)=A_\psi$.

Finally $B_\alpha\to C$. For a base of the product uniform structure on F is formed by the sets $\{(x, y): (p_\phi(x), p_\phi(y))\in U_\phi\}$ as U_ϕ runs through a base for the uniform structure on F_ϕ and ϕ runs through the finite subsets of Γ. Hence a base for the Hausdorff uniform structure on the set of closed nonempty subsets of F is formed by the sets

$$\{(A, B): p_\phi(A) \subseteq U_\phi(p_\phi(B)) \text{ and } p_\phi(B) \subseteq U_\phi(p_\phi(A))\}.$$

Thus $B_\alpha\to C$ if and only if $p_\phi(B_\alpha)\to p_\phi(C)$ in F_ϕ for each finite $\phi\subseteq\Gamma$. But $p_\phi(B_\alpha)\to A_\phi=p_\phi(C)$ and so $B_\alpha\to C$.

Since E is given to be complete, it is a closed subspace of F; each B_α lies in E and therefore so does C. Thus the theorem is proved.

Given a uniform space E with uniform structure ξ, we shall say that another uniform structure η on E is *associated* with ξ if ξ has a base consisting of subsets of $E\times E$ closed in the topology determined by η. This condition insures that ξ can be extended to the η-completion F of E. For if $\mathfrak{U}$ is such a base for ξ, the η-closures in $F\times F$ of the sets of $\mathfrak{U}$ form a base for a uniform structure on F which induces ξ on E. For simplicity, we shall continue to use ξ and η for the extensions of these structures to F.

THEOREM 2. *Let E be a complete separated uniform space with uniform structure ξ and let η be a coarser separated uniform structure on E associated with ξ. Then the set of nonempty η-compact subsets of E is complete under the Hausdorff uniform structure corresponding to ξ.*

PROOF. We may suppose ξ and η extended to the η-completion F of E. Let $\tilde{\xi}$ and $\tilde{\eta}$ be the corresponding Hausdorff uniform structures on the set $\mathfrak{C}$ of η-compact (and therefore also ξ-closed) subsets of F. If (B_α) is a $\tilde{\xi}$-Cauchy net of η-compact subsets of E, (B_α) is also Cauchy in $\mathfrak{C}$ under the coarser structure $\tilde{\eta}$ and so, by Theorem 1, is $\tilde{\eta}$-convergent to some η-compact subset B of F. Now let $\mathfrak{U}$ and $\mathfrak{V}$ be bases for ξ and η respectively, with the sets of $\mathfrak{U}$ η-closed. If $U\in\mathfrak{U}$ is given, there is an α_0 with $B_\alpha\subseteq U(B_\beta)$ for all $\alpha, \beta\geqq\alpha_0$. Then for each $V\in\mathfrak{V}$ there is an $\alpha(V)\geqq\alpha_0$ with $B_\alpha\subseteq V(B)$ and $B\subseteq V(B_\alpha)$ for all $\alpha\geqq\alpha(V)$. Thus, if $\beta\geqq\alpha_0$, $B\subseteq V(B_\alpha)\subseteq V(U(B_\beta))$ for a suitable α. Hence

$$B\subseteq \bigcap\{V(U(B_\beta)):\ V\in\mathfrak{V}\}, \text{ and similarly}$$

$$B_\beta \subseteq \bigcap \{U(V(B)) : V \in \mathcal{V}\}.$$

If we show that this implies that $B \subseteq U(B_\beta)$ and $B_\beta \subseteq U(B)$ we shall have proved that (B_α) converges to B in ξ; since E is a ξ-closed subset of F, it will then follow that $B \subseteq E$ and the theorem will be proved. It will be sufficient to show that, if A is η-compact,

$$\bigcap \{V(U(V(A))) : V \in \mathcal{V}\} \subseteq U(A).$$

If y belongs to the left side of this, there are points $x_V \in A$ with $(x_V, y) \in VUV$. Let x be an η-adherent point of $(x_V : V \in \mathcal{V})$; then $x \in A$ and to each V corresponds a $V' \subseteq V$ with $(x, x_{V'}) \in V$. Thus $(x, y) \in V'UV'V \subseteq VVUVV$ for all $V \in \mathcal{V}$. Since U is η-closed, $(x, y) \in U$ and $y \in U(A)$, as required.

COROLLARY. *In the space of nonempty ξ-closed subsets of E with the Hausdorff uniform structure corresponding to ξ, the η-relatively compact subsets of E form a closed subspace.*

PROOF. Suppose that each B_α is η-relatively compact and that $B_\alpha \rightarrow B$ under ξ. If $U \in \mathcal{U}$, there is an α_0 such that, for all $\alpha, \beta \geqq \alpha_0$, $B_\alpha \subseteq U(B_\beta)$. Hence, if bars denote η-closures, $\overline{B_\alpha} \subseteq V(U(B_\beta)) \subseteq V(U(\overline{B_\beta}))$ for all $V \in \mathcal{V}$. Because $\overline{B_\beta}$ is η-compact, it follows as in the theorem that $\overline{B_\alpha} \subseteq U(\overline{B_\beta})$, so that $(\overline{B_\alpha})$ is ξ-Cauchy. Hence $(\overline{B_\alpha})$ is ξ-convergent to the η-compact set C, say. Now, if $V \in \mathcal{V}$, we have, for sufficiently large α, since ξ is finer than η,

$$B \subseteq V(B_\alpha) \subseteq V(\overline{B_\alpha}) \subseteq V(V(C)).$$

Hence $B \subseteq C$ and B is η-relatively compact.

The space of η-relatively compact subsets of E need not be complete; there is a counterexample later. The corollary has an immediate application to function spaces.

THEOREM 3. *Let S be a set and $\mathfrak{A}$ a family of subsets. Also let E be a complete separated uniform space with uniform structure ξ and let η be a coarser separated uniform structure on E associated with ξ. If F is the space of mappings of S into E which take the sets of $\mathfrak{A}$ onto η-relatively compact subsets of E, then F is complete under the uniform structure of ξ-uniform convergence on the sets of $\mathfrak{A}$.*

PROOF. Let (f_α) be a Cauchy net in F. Since the space E^S of all mappings of S into E is complete under the $(\mathfrak{A}, \xi)$ uniform structure, (f_α) converges in E^S to f, say. It follows that, for each $A \in \mathfrak{A}$, the ξ-closure of $f_\alpha(A)$ converges to the ξ-closure of $f(A)$ under the Hausdorff uniform structure corresponding to ξ. Since each $f_\alpha(A)$ is η-relatively compact, the corollary of Theorem 2 shows that $f(A)$ is also η-relatively compact. Hence $f \in F$ and F is complete.

The motive for considering associated uniform structures comes from the theory of locally convex spaces. Here we may speak about topologies instead of uniform structures. Let ξ and η be two separated locally convex topologies on a vector space E. Then (the uniform structure determined by) η is associated with (that determined by) ξ if and only if there is a base of η-closed ξ-neighbourhoods of the origin. An immediate example, the one which suggested the terminology, is obtained by taking for η the associated weak topology corresponding to ξ. The two topologies ξ and η are each associated with the other if and only if there is a vector space F is duality with E such that each of ξ and η is a topology of uniform convergence on a family of $\sigma(F, E)$-bounded subsets whose union spans F. (For if this last condition is satisfied there is a base of ξ-neighbourhoods of the origin consisting of polars of subsets of F; they are therefore $\sigma(E, F)$-closed and so closed in the finer topology η. Conversely, if ξ and η are associated, we can take for F the intersection of the duals E'_ξ and E'_η. If U is an η-closed ξ-neighbourhood of the origin, which we may suppose absolutely convex, its polar in E'_η is $U^\circ \cap F$ and so the polar of $U^\circ \cap F$ in E is the η-closure of U, which is U itself. Thus ξ is the topology of uniform convergence on sets of the form $U^\circ \cap F$, and similarly for η.)

The natural analogue of Theorem 3 for linear spaces is the following, obtained at once from Theorem 3. Let S be a separated locally convex space, and E a complete separated locally convex space with topology ξ. Also let η be a coarser separated topology on E associated with ξ. Then the space of linear mappings of S into E which take bounded sets of S onto η-relatively compact subsets of E is complete under the topology of ξ-uniform convergence on the bounded subsets of S.

We make one or two additional comments on Theorem 2, when E is a locally convex space. It is immediate that, with the notations used in the theorem, every $\tilde{\eta}$-complete set of η-compact sets is also $\tilde{\xi}$-complete. A proof that closely parallels that of Theorem 2 shows that, if E has the same dual under ξ and η, every $\tilde{\eta}$-complete set of closed convex subsets of E is also $\tilde{\xi}$-complete. (Compare the corresponding result for points: Bourbaki [**2**, Chapter I, §1, Proposition 8].)

In a complete separated locally convex space, although the weakly compact subsets form a complete space, the space of closed bounded subsets may fail to be complete. We shall give an example of a nonconvergent Cauchy net of closed bounded subsets of a product E of uncountably many copies of the Banach space $\boldsymbol{m}$. This will also furnish the counterexample promised after the corollary to Theorem 2. For if ξ is the product of the norm topologies and η

the (associated) product of the topologies $\sigma(m, l^1)$, the sets of the net will be η-relatively compact, since each bounded set in m is $\sigma(m, l^1)$-relatively compact.

Let Γ be an uncountable set of indices and put $E = \times\{E_\gamma : \gamma \in \Gamma\}$, where, for each $\gamma \in \Gamma$, E_γ is the Banach space m. On E, we consider the product of the norm topologies. We shall say that a subset A of m has the property (*) if every element of A is a sequence of nonnegative numbers converging to zero and the sum s_B of the elements in each (nonempty) finite subset B of A has norm in the range $3/4 \leqq \|s_B\| \leqq 1$. There are such sets; if e_n denotes the sequence whose nth term is 1 and all of whose other terms are 0, any finite or infinite subset of $\{e_n : n = 1, 2, \cdots\}$ has the property (*). But if A has the property (*), then A is finite or countable, because, for each n, there is at most one sequence in A whose nth term is greater than or equal to 3/4, and each sequence in A has at least one such term. Now, for each finite subset ϕ of Γ let B_ϕ be the subset of E consisting of those points $x = (x_\gamma)$ for which $\{x_\gamma : \gamma \in \phi\}$, regarded as a subset of m, has the property (*), and $\|x_\gamma\| \leqq 1$ for all $\gamma \in \Gamma$. Then each B_ϕ is bounded in E, and is closed, being an intersection of closed sets. We show now that, if p_ϕ is the natural projection of E onto $\times\{E_\gamma : \gamma \in \phi\}$, $p_\phi(B_\psi) = p_\phi(B_\phi)$ for all $\phi \subseteq \psi$. Since the sets B_ψ decrease with increasing ψ we need only prove that if $\{x_\gamma : \gamma \in \phi\}$ has the property (*), points $\{x_\gamma : \gamma \notin \phi\}$ can be so defined that $(x_\gamma : \gamma \in \Gamma) \in B_\psi$. If ϕ has r elements, we can choose k so large that the nth coordinate of x_γ is less than $(4r)^{-1}$ for all $n \geqq k$ and all $\gamma \in \phi$. Then for each γ in ψ but not in ϕ we choose distinct integers $n(\gamma) \geqq k$ and put $x_\gamma = (3/4)e_{n(\gamma)}$. Finally for $\gamma \notin \psi$ we put $x_\gamma = o$. It is then a routine matter to check that this definition will suffice. Because $p_\phi(B_\psi)$ is constant for all ψ with $\phi \subseteq \psi$, (B_ψ) is a Cauchy net. Also, since (B_ψ) is decreasing, if it converges, it must converge to the intersection of the sets B_ψ. But this intersection is empty, for if $x = (x_\gamma)$ were a point in it, the uncountable set $\{x_\gamma : \gamma \in \Gamma\}$ would have the property (*). Hence (B_ψ) does not converge.

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University of Kansas