SOME REMARKS ON NILPOTENT GROUPS WITH ROOTS

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1. Let ϖ be a nonempty set of primes. Then a group G is called an $E\varpi$-group if the equation

$$x^p = g \tag{1}$$

is soluble for every $p \in \varpi$ and every $g \in G$. Similarly we call a group G a $D\varpi$-group if the equation (1) (above) is not only soluble, but *uniquely soluble*. The class of $E\varpi$-groups and the class of $D\varpi$-groups will be denoted respectively by $E\varpi$ and $D\varpi$. Clearly $E\varpi \supset D\varpi$. There is a less trivial relationship between these two classes of groups. If we denote by $\Theta(C)$ the class of groups which occur as homomorphic images of the members of a given class of groups C, then $\Theta(D\varpi) = E\varpi$ (see G. Baumslag [1]).

Now suppose that N denotes the class of nilpotent groups. One of the consequences of our results in this note is the fact that $\Theta(D\varpi \cap N) = E\varpi \cap N$. However we shall show that if L is the class of locally nilpotent groups then $\Theta(D\varpi \cap L) \neq E\varpi \cap L$. This leads us to ask whether, perhaps, $\Theta(D\varpi \cap S) \supseteq E\varpi \cap L$, where S here is the class of locally soluble groups.

2. Let G be a nilpotent group of class c. The subset X of G is said to *freely generate* G if

(i) X generates G, and

(ii) for every group H which is nilpotent of class at most c and every mapping θ of X into H there exists a homomorphism ϕ of G into H which coincides with θ on X. A nilpotent group of class c which is freely generated by some set is called a free nilpotent group. These groups are simply isomorphic copies of F/F_{c+1}, where F is some free group and F_{c+1} is the $(c+1)$st term of the lower central series of F.

We shall embed a free nilpotent group G in a nilpotent $D\varpi$-group G^*. This embedding has already been carried out in a rather more general context by A. I. Mal′cev [5]; however he makes use of Lie ring methods and so we prefer to carry out here a different direct embedding procedure which lends itself more easily to the application we have in mind.

Suppose then that G is a free nilpotent group of class c freely generated by the set X. Put $G = G_1$ and $X = X_1$. Suppose now that for each

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positive integer n G_n is a free nilpotent group freely generated by X_n, with

$$|X_n| = |X_{n-1}| \qquad (n = 2, 3, \cdots), \tag{2}$$

where $|S|$ denotes the cardinality of the set S. Let

$$p_1, p_2, \cdots, p_n, \cdots \tag{3}$$

be an infinite sequence of primes in ϖ, chosen so that for any $p \in \varpi$ and any positive integer N there exists an integer $M \geqq N$ such that $p_M = p$. Now define

$$H_n = gp(x^{p_{n-1}}; x \in X_n) \qquad (n \geqq 2).$$

By a theorem of G. Baumslag [2], H_n is nilpotent of class c and is freely generated by

$$X_n^{p_{n-1}} = \{x^{p_{n-1}} \mid x \in X_n\}.$$

Clearly $|X_n^{p_{n-1}}| = |X_n| = |X_{n-1}|$. Hence we may identify G_{n-1} with H_n for $n = 2, 3, \cdots$. Finally we put

$$G^* = \bigcup_{n=1}^{\infty} G_n.$$

Clearly G^* is nilpotent of class c. By the choice of the sequence (3) every element of G^* can be written as a product of pth powers for any $p \in \varpi$. Consequently, by a theorem of S. N. Černikov (see Kuroš [4, vol. 2, p. 238]), *every* element of G^* has a pth root for each $p \in \varpi$. Moreover G^* is clearly torsion-free, since each G_n has this property. Therefore as extraction of pth roots in a torsion-free nilpotent group is unique whenever it is possible (A. I. Mal′cev, see e.g. Kuroš [4, vol. 2, p. 247]), G^* is a $D\varpi$-group. In the following theorem we adopt the notation of this paragraph.

THEOREM 1. *Suppose σ is a set of primes such that $\sigma \geqq \omega$. Then for every E_σ-group H which is nilpotent of class at most c and every mapping θ of X into H there exists a homomorphism ϕ of G^* into H which coincides with θ on X.*

PROOF. Since G is freely generated by X we can extend θ to a homomorphism θ_1 of G into H. Suppose, inductively, that θ_i is a homomorphism of G_i into H. Now θ_i is determined by its effect on X_i, the free set of generators of G_i. Suppose $y \in X_i$. Then, by our construction of G^*,

$$y = x^{p_i} \qquad (x \in X_{i+1}).$$

We put

$$x\theta_{i+1} = z,$$

where z is chosen to be any p_ith root of $y\theta_i$. Thus

$$(x\theta_{i+1})^{p_i} = z^{p_i} = y\theta_i. \tag{4}$$

In this way we define the effect of a mapping θ_{i+1} on the free generators X_{i+1} of G_{i+1} and, since X_{i+1} is a free set of generators, θ_{i+1} can be extended to a homomorphism of G_{i+1} into H, which we again denote by θ_{i+1}. It follows from (4) that θ_{i+1} extends θ_i. Finally we define

$$\phi = \bigcup_{i=1}^{\infty} \theta_i;$$

i.e., if $g \in G^*$, then we define $g\phi = g\theta_n$ if $g \in G_n$. Clearly ϕ is a homomorphism of G^* into H. Furthermore, by its very definition ϕ coincides with θ on X. This completes the proof.

COROLLARY. $\Theta(D\varpi \cap N) = E\varpi \cap N$.

PROOF. Let $H \in E\varpi \cap N$. Then H is nilpotent of, say, class c. Choose X so that

$$|X| = |H| \tag{5}$$

and let G^* be as above. Define θ to be any mapping from X onto H, whose existence is assured by (5). By Theorem 1, this mapping can be extended to a homomorphism of G^* onto H. The corollary therefore follows.

It is worth pointing out that the two classes of groups $D\varpi \cap N_c$, $E\varpi \cap N_c$, where N_c is the class of groups which are nilpotent of class at most c, both form varieties of algebras in the sense of P. Hall; in other words they are equationally definable. The group G^* is simply one of the free algebras of the variety $D\varpi \cap N_c$. Thus, on completely general grounds, any mapping of X into a group H *which belongs* to $D\varpi \cap N_c$, can be extended to a homomorphism of G^* into H. This statement, however, does not immediately imply the corresponding one if $H \in E\varpi \cap N_c$.

Let us call an integer $m > 0$ an ϖ-number if the prime divisors of m belong to ϖ. The following theorem then holds.

THEOREM 2. *Let $G \in D\varpi \cap L$, i.e., suppose G is a locally nilpotent* *group in D_ϖ, and let N be a normal subgroup of G. Then the elements in G/N whose orders are ϖ-numbers form a subgroup contained in the center of G/N.*

PROOF. It is well known that in a locally nilpotent group the elements whose orders are ϖ-numbers form a subgroup (cf., e.g., Kuroš [4, vol. 2, p. 216]). It remains to prove, therefore, that if $g^m \in N$, where $g \in G$ and m is an ϖ-number, then $[g, h] \in N$ for every $h \in G$.

Since G is locally nilpotent, g and h generate a nilpotent subgroup of G. Let H be a subgroup of G containing g and h which is nilpotent and also belongs to D_ϖ—the existence of such a subgroup H follows e.g. by a theorem of G. Baumslag [1]. Since

$$HN/N \cong H/H \cap N,$$

HN/N is nilpotent. Clearly HN/N then belongs to E_ϖ, since $H \in E_\varpi$. Now in a nilpotent group in E_ϖ the elements whose orders are ϖ-numbers belong to the centre (see [1]). In particular gN belongs to the centre of HN/N, i.e., $[h, g] \in N$. Since h was arbitrary, the theorem follows.

Now every finite p-group can be embedded in a locally nilpotent E_ϖ-group (see G. Baumslag [3]). Consequently there are nonabelian locally nilpotent E_ϖ-groups which are periodic. This would, however, not be the case if $\Theta(D_\varpi \cap L) = E_\varpi \cap L$, by Theorem 2. Thus we have proved

THEOREM 3. $\Theta(D_\varpi \cap L) \neq E_\varpi \cap L$.

One is tempted to enquire, at this point, whether there is a class C of groups with C properly contained in D_ϖ, such that

$$\Theta(C) \supseteq E_\varpi \cap L.$$

3. It is perhaps of interest to make some further remarks concerning the group G^* of §2. Let us call a subgroup of G^* which is at the same time a D_ϖ-group, an ϖ-subgroup. We say that a subset Y of G^* is a free ϖ-generating set of G^* if

(i) Y ϖ-generates G^*, i.e., the smallest ϖ-subgroup of G^* containing Y is G^* itself, and

(ii) for every D_ϖ-group H which is nilpotent of class at most c, and every mapping θ of Y into H, there exists a mapping ϕ of G^* into H which coincides with θ on Y.

The following theorem then holds (again we assume the notation of §2).

THEOREM 4. *Let Y be a subset of G^*. Suppose $|X| = |Y|$ and that $|X|$ is finite. If Y ϖ-generates G^*, then Y is a free ϖ-generating set of G^*. Further, if $F = gp(Y)$, then F is a free nilpotent group of class c which is freely generated by Y.*

We need the following lemma for the proof of Theorem 4.

LEMMA. *Let A be a nilpotent group of class c which can be generated by $s(<\infty)$ elements. Suppose that B is a subgroup of A which is nilpotent of class c and is freely generated by $\{b_1, b_2, \cdots, b_s\}$. Then A is a free nilpotent group.*

PROOF. Suppose $A = gp(a_1, a_2, \cdots, a_s)$. Let θ_1 be the homomorphism of B onto A defined by

$$b_1\theta_1 = a_1, b_2\theta_1 = a_2, \cdots, b_s\theta_1 = a_s.$$

Let N_1 be the kernel of θ_1:

$$B/N_1 \cong A.$$

If $N_1 = 1$, then $A \cong B$ as required. If $N_1 \neq 1$, then by first carrying out the same procedure with B/N_1, which contains a free nilpotent subgroup isomorphic to A, and then iterating this process, we obtain a series

$$1 < N_1 < N_2 < \cdots$$

which is defined in the obvious way; thus we have a properly ascending series of subgroups of the finitely generated nilpotent group A. This is impossible by a theorem of K. A. Hirsch (see e.g., Kuroš [**4**, vol. 2, p. 193]) and so the proof of the lemma is complete.

To prove Theorem 4, we begin by noting (see Kuroš [**4**, vol. 2, p. 249]) that as Y ϖ-generates G^*, there exists an ϖ-number m such that $x^m \in F$ for each $x \in X$. Thus, by the lemma, F is a free nilpotent group of class c freely generated by Y. We put $Y_1 = Y$, $F_1 = F$ and having defined Y_n so that Y_n freely generates $F_n = gp(Y_n)$, we define (see (3)):

$$Y_{n+1} = \{x \mid x^{p_n} = y \text{ for some } y \in Y_n\}.$$

Then $|Y_{n+1}| = |Y|$. Furthermore, if we put $F_{n+1} = gp(Y_{n+1})$, by the lemma, it follows that F_{n+1} is free nilpotent of class c freely generated by Y_{n+1}. Finally we put $F^* = \bigcup_{i=1}^{\infty} F_i$; then by the choice of the sequence (3), every element of F^* is a product of pth powers for every $p \in \varpi$. So, invoking Černikov's theorem again, F^* is a $D\varpi$-group. So $F^* = G^*$. It follows now by its very construction that $F^*(=G^*)$ is freely ϖ-generated by Y (cf. the proof of Theorem 1). So the proof of Theorem 4 is complete.

We remark, in conclusion, that the lower central series of the group G^* has a quite analogous structure to that of the lower central series of a free nilpotent group of rank $|X|$. The details involve the use of basic commutators and so, in order to keep this note self-contained, we prefer to omit the proof of this statement until a later date.

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