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SOLUTION TO A PROBLEM OF ROSE AND ROSSER

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In a recent article by Rose and Rosser [1], the question is raised concerning the possibility of proving the following theorem using only the first three of Łukasiewicz' axioms for infinite-valued logic together with his rules of inference [2]:

$$(3.51) \qquad CCQPCQR \equiv CCPQCPR.$$

The question is not only interesting in itself, but sheds some light on problems of independence relating to Łukasiewicz' axioms. For example, in another recent paper [3], C. A. Meredith establishes the dependence of Łukasiewicz' fourth axiom, using only the first three of Łukasiewicz' axioms together with Rose and Rosser's Theorem 3.51.

The purpose of this paper is to establish a negative answer to the Rose-Rosser question. This will be done in a way which will illustrate the use of many-valued logics [4] as instruments for deciding questions of independence, and from this, one will be able to see that in deciding a negative answer to the Rose-Rosser question, a logic with at least four truth-values is required. To this end, let APQ be defined as $CCPQQ$ and consider the following axiom schemes and rule of inference:

Axiom schemes:

A1. $CPCQP$.
A2. $CCPQCCQRCPR$.
A3. $CAPQAQP$.

Rule of inference:

R1. If P and CPQ, then Q.

A1, A2, and A3 are the first three axioms introduced by Łuka-siewicz for infinite-valued logic, and since axiom schemes are being used, there is no need for a rule of substitution in addition to R1. A yields sign "⊢" may now be defined in the usual manner in terms of A1, A2, A3, and R1. Let $c(p, q)$ denote the truth-value function which is associated with a statement of the form CPQ, and let the values of $c(p, q)$ be defined by the following matrix:

$c(p, q)$	1	2	3	4
1	1	2	3	4
2	1	1	1	3
3	1	3	1	3
4	1	3	1	1

(T)

By means of the matrix T, the axiomatic system based on A1, A2, A3, and R1 may be interpreted as a 4-valued statement calculus with one designated truth-value in the sense of Rosser and Turquette. (See Chapters II and III of *Many-valued logics*.) It is then easy to establish the following results:

THEOREM 1. *If the designated truth-value is* 1 *and* ⊢X, *then* X *takes the designated truth-value exclusively—i.e., the axiomatic stipulation based on* A1, A2, A3, *and* R1 *is plausible with respect to the truth-value stipulation determined by* $M = 4$, $S = 1$, *and the matrix* T *(see Many-valued logics, pp. 27–28 and p. 34).*

PROOF. By applying a standard truth-table method of calculation making use of the matrix T, it can be shown that each of the axioms A1, A2, A3 takes the truth-value 1 exclusively, and that the rule of inference R1 is valid.

THEOREM 2. *Not*—⊢$ACPQCQP$—*i.e., Łukasiewicz' fourth axiom for infinite-valued logic is not provable using* A1, A2, A3, *and* R1 *alone.*

PROOF. By definition, $ACPQCQP$ is $CCCPQCQPCQP$. A truth-table check by means of T will show that this axiom takes the un-designated value 3 when P takes the value 2 and Q takes the value 4. Hence, Theorem 2 follows from Theorem 1.

THEOREM 3. *Rose and Rosser's* 3.51 *is not provable using only* A1, A2, A3, *and* R1—*i.e., the answer to the Rose-Rosser question is negative.*

PROOF. If 3.51 were provable using only A1, A2, A3 and R1, then by Meredith's proof mentioned above which establishes the dependence of Łukasiewicz' fourth axiom for infinite-valued logic, it would be possible to get $\vdash ACPQCQP$. Since this is contradictory to Theorem 2, Theorem 3 follows at once.

When the present author first solved the Rose-Rosser problem, use was made of a 6-valued logic with two designated truth-values to establish Theorems 1, 2, and 3. The 6-valued logic was based on a 6 by 6 matrix very similar in structure to T. The solution was communicated to Professor Rosser who coded and checked the most difficult truth-table calculations for the 6-valued logic on an electronic computer. After verifying the desired results for 6-valued logic, Rosser suggested that it should be possible to use a 4 by 4 matrix. The choice of the matrix T is in line with his suggestion. That no further such reduction can be made in the number of truth-values for the logic used in establishing Theorems 1, 2, and 3, may be seen by checking all possible 3 by 3 and 2 by 2 matrices. Fortunately, the number of cases required for such a check can be greatly reduced by capitalizing on the requirements specified in the proofs of Theorems 1 and 2.

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