

ON SOME CONDITIONS FOR NONVANISHING OF DETERMINANTS

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1. Among the criteria for nonvanishing of the determinant D of the matrix $A(a_{\mu\nu})(\mu, \nu = 1, \dots, n)$, those of particular interest are the ones which use only the *moduli* $\alpha_{\mu\nu} = |a_{\mu\nu}|$ of the elements and some simple combinations of these moduli. The most famous is given by the so-called Hadamard's Theorem² that $D \neq 0$ if we have

$$(1) \quad \alpha_{\mu\mu} > \sum_{\nu \neq \mu} \alpha_{\mu\nu} \quad (\mu = 1, \dots, n).$$

Later we gave different criteria of this type using in particular the expressions

$$(2) \quad R_\mu = \sum_{\kappa \neq \mu} \alpha_{\mu\kappa}, \quad C_\mu = \sum_{\kappa \neq \mu} \alpha_{\kappa\mu} \quad (\mu = 1, \dots, n),$$

$$(3) \quad R_{\mu,s} = \left(\sum_{\kappa \neq \mu} \alpha_{\mu\kappa}^s \right)^{1/s}, \quad C_{\mu,s} = \left(\sum_{\kappa \neq \mu} \alpha_{\kappa\mu}^s \right)^{1/s} \quad (\mu = 1, \dots, n),$$

$$(4) \quad m_\mu = \text{Max}_{\kappa \neq \mu} \alpha_{\mu\kappa} = R_{\mu,\infty}, \quad m'_\mu = \text{Max}_{\kappa \neq \mu} \alpha_{\kappa\mu} = C_{\mu,\infty} \quad (\mu = 1, \dots, n).$$

2. An essential link towards the main result of this note was given in 1951 in a previous communication,³ namely that for any fixed $\alpha, 0 < \alpha < 1$, the inequalities

$$R_\mu^\alpha C_\mu^{1-\alpha} < \alpha_{\mu\mu}, \quad (\mu = 1, \dots, n)$$

are sufficient. Some special cases of this with $\alpha = 1/2$, were found earlier by W. V. Parker, 1937, and E. W. Barankin, 1945.

The most general criterion using R_μ and C_μ was given by the inequalities

$$(5) \quad R_\mu^\alpha C_\mu^{1-\alpha} R_\nu^\alpha C_\nu^{1-\alpha} < \alpha_{\mu\mu} \alpha_{\nu\nu} \quad (\mu \neq \nu; \mu, \nu = 1, \dots, n),$$

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² See the note of Olga Taussky-Todd, *A recurring theorem on determinants*, Amer. Math. Monthly vol. 56 (1949) pp. 672-676, which contains a very complete bibliography of this theorem. A fairly complete treatment of the criteria of the type considered here can be found in the book: M. Parodi, *La localisation des valeurs caractéristiques des matrices et ses applications*, Paris, Gauthier-Villars, 1959.

³ A. M. Ostrowski, *Ueber das Nichtverschwinden einer Klasse von Determinanten und die Lokalisierung der charakteristischen Wurzeln von Matrizen*, Compositio Math. vol. 9 (1951). This paper will be quoted in the following as U.

where α is an arbitrarily chosen fixed number with

$$(6) \quad 0 \leq \alpha \leq 1.^3$$

As to $R_{\mu,s}$ and $C_{\mu,s}$, the corresponding criterion is

$$(7) \quad \sum_{\mu=1}^n \frac{1}{1 + \left(\frac{\alpha_{\mu\mu}}{R_{\mu,p}} \right)^q} < 1$$

for a fixed but arbitrary choice of p and q satisfying

$$(8) \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p \geq 1, \quad q \geq 1,$$

and, of course, the criterion obtained from (7) by replacing $R_{\mu,p}$ by $C_{\mu,p}$. For $q=1$, we have $p=\infty$, and (7) becomes

$$(9) \quad \sum_{\mu=1}^n \frac{1}{1 + \alpha_{\mu\mu}/m_{\mu}} < 1.$$

We see in particular that in the criterion (7) only $R_{\mu,s}$ with $s \geq 1$ are used.⁴

3. In this paper we resume the method of U, introducing now a new parameter into the criteria using the expressions (3). Our essential result is that $D \neq 0$ if we have

$$(10) \quad R_{\mu,\alpha p}^{\alpha} C_{\mu,(1-\alpha)q}^{1-\alpha} < \alpha_{\mu\mu} \quad (\mu = 1, \dots, n)$$

for fixed but arbitrarily chosen α with $0 < \alpha < 1$ and p and q satisfying (6). More generally, we have the sufficient condition for $D \neq 0$ in

$$(11) \quad R_{\mu,\alpha p}^{\alpha} C_{\mu,(1-\alpha)q}^{1-\alpha} R_{\nu,\alpha p}^{\alpha} C_{\nu,(1-\alpha)q}^{1-\alpha} < \alpha_{\mu\mu} \alpha_{\nu\nu} \quad (\mu \neq \nu; \mu, \nu = 1, \dots, n).$$

For $p=1/\alpha$ we obtain from (10) and (11) the criteria derived in U, which are of course less complicated than (10) and (11). On the other hand, using other specialization of parameters, we obtain new types of simple criteria.

4. Taking for $0 < \alpha < 1$, $q=1$, $p=\infty$, we obtain from (10) the conditions

$$m_{\mu}^{1-\alpha} C_{\mu,\alpha}^{\alpha} < \alpha_{\mu\mu} \quad (\mu = 1, \dots, n),$$

that is,

⁴ The criteria (7) and (9) have been derived in A. M. Ostrowski, *Sur les conditions générales pour la régularité des matrices*, Rend. Mat. e Appl. ser. V, vol. X (1951) pp. 156-168.

$$(12) \quad m_\mu^{1-\alpha} \sum_{\kappa \neq \mu} \alpha_{\kappa\mu}^\alpha < \alpha_{\mu\mu} \quad (\mu = 1, \dots, n),$$

and of course the corresponding conditions obtained by using A' instead of A .

5. The essentially new part of the proof is that of the conditions (10), which we give in §§6, 7 as the proof of Theorem I. As to further discussions, they are in most cases exactly the same as in U and we will usually be able to deal with the corresponding parts of the argument by simply referring to U.

6. THEOREM I. D does not vanish if we have, for an α with $0 < \alpha < 1$ and a couple p, q satisfying (8), the inequalities (10).

PROOF. Without loss of generality we can assume that $q < \infty$, since otherwise it would be sufficient to apply the discussion to A' .

If we had $D=0$, we would have a nontrivial solution (x_1, \dots, x_n) of the system

$$(13) \quad \sum_{\nu=1}^n \alpha_{\mu\nu} x_\nu = 0 \quad (\mu = 1, \dots, n).$$

Put

$$(14) \quad |a_{\mu\nu}| = \alpha_{\mu\nu}, \quad |x_\nu| = \xi_\nu \quad (\mu, \nu = 1, \dots, n).$$

Then, isolating in (13) the term with x_μ and going over to the moduli, we obtain

$$(15) \quad \alpha_{\mu\mu} \xi_\mu \leq \sum_{\nu \neq \mu} \alpha_{\mu\nu} \xi_\nu \quad (\mu = 1, \dots, n);$$

this can be written in the form

$$(16) \quad \alpha_{\mu\mu} \xi_\mu \leq \sum_{\nu \neq \mu} \alpha_{\mu\nu}^\alpha (\alpha_{\mu\nu}^{1-\alpha} \xi_\nu) \quad (\mu = 1, \dots, n).$$

Applying to (16) Hölder's inequality, we obtain

$$\alpha_{\mu\mu} \xi_\mu \leq \left(\sum_{\nu \neq \mu} \alpha_{\mu\nu}^{\alpha p} \right)^{1/p} \left(\sum_{\nu \neq \mu} \alpha_{\mu\nu}^{(1-\alpha)q} \xi_\nu^q \right)^{1/q} \quad (\mu = 1, \dots, n),$$

or using (3),

$$(17) \quad \alpha_{\mu\mu} \xi_\mu \leq R_{\mu, \alpha p}^\alpha \left(\sum_{\nu \neq \mu} \alpha_{\mu\nu}^{(1-\alpha)q} \xi_\nu^q \right)^{1/q} \quad (\mu = 1, \dots, n).$$

7. Raising (17) to the q th power, we can write

$$R_{\mu, \alpha p}^{-\alpha q} \alpha_{\mu \mu}^q \xi_{\mu}^q \leq \sum_{\nu \neq \mu} \alpha_{\mu \nu}^{(1-\alpha)q} \xi_{\nu}^q \quad (\mu = 1, \dots, n).$$

Summing this over $\mu = 1, \dots, n$, we obtain

$$\sum_{\mu=1}^n R_{\mu, \alpha p}^{-\alpha q} \alpha_{\mu \mu}^q \xi_{\mu}^q \leq \sum_{\mu=1}^n \sum_{\nu \neq \mu} \alpha_{\mu \nu}^{(1-\alpha)q} \xi_{\nu}^q.$$

But, if we interchange the order of summation in the right-hand sum and use (3), we obtain

$$\sum_{\nu=1}^n \xi_{\nu}^q \sum_{\mu \neq \nu} \alpha_{\mu \nu}^{(1-\alpha)q} = \sum_{\nu=1}^n C_{\nu, (1-\alpha)q}^{(1-\alpha)q} \xi_{\nu}^q = \sum_{\mu=1}^n C_{\mu, (1-\alpha)q}^{(1-\alpha)q} \xi_{\mu}^q,$$

and we get

$$(18) \quad \sum_{\mu=1}^n R_{\mu, \alpha p}^{-\alpha q} \alpha_{\mu \mu}^q \xi_{\mu}^q \leq \sum_{\mu=1}^n C_{\mu, (1-\alpha)q}^{(1-\alpha)q} \xi_{\mu}^q.$$

But, since at least one of ξ_{μ}^q is unequal to zero, (18) cannot hold if we have (10). Theorem I is proved.

8. As in U, we can replace the right-hand expressions in the inequalities (10) by

$$(19) \quad \alpha R_{\mu, \alpha p} + (1 - \alpha) C_{\mu, (1-\alpha)q},$$

as we have generally

$$(20) \quad \alpha x + (1 - \alpha)y \geq x^{\alpha} y^{1-\alpha} \quad (x > 0, y > 0, 0 \leq \alpha \leq 1).$$

In this way we obtain a weaker formulation of Theorem 1, which is, however, sometimes easier to apply numerically.

From Theorem I follows immediately

THEOREM II. *D does not vanish if we have for an α with $0 < \alpha < 1$ and a couple p, q satisfying (8) the inequalities (11).*

The proof is literally the same as in U, §19, if we introduce the expressions

$$(21) \quad s_{\mu} = \frac{1}{\alpha_{\mu \mu}} R_{\mu, \alpha p}^{\alpha} C_{\mu, (1-\alpha)q}^{1-\alpha}.$$

9. An immediate consequence of Theorem 1 is

THEOREM III. *For any α with (6) and any couple p, q satisfying (8) each fundamental root of A lies inside or on one of the circles*

$$(22) \quad |\lambda - \alpha_{\mu\mu}| \leq R_{\mu, \alpha p}^\alpha C_{\mu, (1-\alpha)q}^{1-\alpha} \quad (\mu = 1, \dots, n).$$

The modulus of each fundamental root of A is at the most equal to

$$(23) \quad \text{Max}_\mu (\alpha_{\mu\mu} + R_{\mu, \alpha p}^\alpha C_{\mu, (1-\alpha)q}^{1-\alpha}) \leq \left(\sum_{\kappa=1}^n \alpha_{\mu\kappa}^\alpha \right)^{1/p} \left(\sum_{\kappa=1}^n \alpha_{\kappa\mu}^{(1-\alpha)q} \right)^{1/q}.$$

The inequality in (23) follows, of course, from Hölder's inequality.

Particularly simple expressions are obtained if we consider the limiting case $q=1$. Then (23) becomes

$$\text{Max}_\mu \left(\alpha_{\mu\mu} + m_\mu^\alpha \sum_{\kappa \neq \mu} \alpha_{\kappa\mu}^{1-\alpha} \right).$$

Define

$$(24) \quad M_\mu = \text{Max}_\mu (\alpha_{\mu\mu}, m_\mu) = \text{Max}_\kappa \alpha_{\mu\kappa}.$$

Then we have as a bound for the moduli of all fundamental roots of A

$$(25) \quad \text{Max}_\mu M_\mu^\alpha \sum_{\kappa=1}^n \alpha_{\kappa\mu}^{1-\alpha}.$$

Similarly, from Theorem II it follows that for any choice of α, p, q satisfying (6) and (8) each fundamental root of A lies inside or on the boundary of one of the ${}_nC_2$ lemniscate-shaped domains

$$(26) \quad |\lambda - \alpha_{\mu\mu}| |\lambda - \alpha_{\nu\nu}| \leq (R_{\mu, \alpha p} R_{\nu, \alpha p})^\alpha (C_{\mu, (1-\alpha)q} C_{\nu, (1-\alpha)q})^{1-\alpha} \quad (\mu \neq \nu).$$

10. In the case of Hadamard's theorem, O. Taussky-Todd generalized the argument to the case where the $>$ in the inequalities (1) are replaced by \geq . The corresponding discussion for the criterion using simultaneously R_μ and C_μ was given in U as Theorems VI and VII and proved in §11–18. Exactly the same argument as in U can be applied to the inequalities (10) and the inequalities using (19). We obtain

THEOREM IV. Assume an α with $0 < \alpha < 1$ and a couple p, q satisfying (8) with $p < \infty, q < \infty$.

If we have for the matrix A the inequalities

$$(27) \quad R_{\mu, \alpha p}^\alpha C_{\mu, (1-\alpha)q}^{1-\alpha} \leq \alpha_{\mu\mu} \quad (\mu = 1, \dots, n),$$

and if $D=0$, then either A is totally reducible or we have in all relations (27) the equality sign and A is irreducible.

If we have

$$(28) \quad \alpha R_{\mu, \alpha p} + (1 - \alpha) C_{\mu, (1-\alpha)q} \leq \alpha_{\mu\mu} \quad (\mu = 1, \dots, n),$$

if the matrix is totally irreducible and if $D=0$, then the equality sign holds in all relations (28), we have for each μ : $R_{\mu, \alpha p} = C_{\mu, (1-\alpha)q}$ and A is simply irreducible. In this case the rows and columns of A can be multiplied by convenient factors of modulus 1 in such a way that after this multiplication the sums of the elements in each row and in each column vanish.⁵

11. It may finally be observed that from the results proved in a previous communication⁶ and from Theorem I and Theorem II follows

THEOREM V. *Under the conditions of Theorem I and Theorem II, the columns of A can be multiplied by convenient positive factors in such a way that after this multiplication the relations (1) hold.*

⁵ A matrix A is *reducible* if by a cogradient permutation of rows and columns it can be brought into the form $\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$; otherwise, it is *irreducible*. A is *totally reducible* if by a cogradient permutation of rows and columns it can be brought into the form $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$; otherwise, A is *totally irreducible*.

⁶ Alexander Ostrowski, *Über die Determinanten mit überwiegender Hauptdiagonale*, Comment. Math. Helv. vol. 10 (1937) pp. 69–96.

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