REGULAR EXTENSIONS AND THE SOLVABILITY OF OPERATOR EQUATIONS

ROGER H. HOMER[1]

Let T_1 be a closed linear operator in a complex Banach space. In this paper we are concerned with the effect of changes in the complex parameter λ on the solvability of the equation $(T_1-\lambda I)x=y$. By introducing the notion of a regular extension we are able to generalize a result of A. S. Markus [2].

In order to introduce extension terminology we consider T_1 to be an extension of a closed linear operator T_0, denoted $T_0\subset T_1$. We use $D(T)$ to denote the domain of an operator T, $R(T)$ for the range of T and $N(T)$ for the null space of T. $K(T)$ denotes the set of all elements y such that the equation $T^n x=y$ is solvable for every positive integer n, a concept originally introduced by F. Riesz [3, p. 87].

We call a closed linear operator T a regular extension at λ if $T_0\subset T\subset T_1$, $R(T-\lambda I)=R(T_1-\lambda I)$ and $T-\lambda I$ has a bounded inverse. We call T a regular extension near λ_0 if for every λ in some neighborhood of λ_0, T is a regular extension at λ. Let Γ be any connected component of the open set consisting of all complex numbers λ such that there exists a regular extension near λ.

THEOREM. *For each element y, either y belongs to $K(T_1-\lambda I)$ for all λ in Γ, or there is no λ in Γ such that y belongs to $K(T_1-\lambda I)$ and the set of all λ in Γ such that y belongs to $R(T_1-\lambda I)$ has no accumulation point in Γ.*

The above mentioned result of A. S. Markus [2] differs from this only in the hypothesis. He takes Γ to be an open connected set such that for every λ in Γ, $R(T_1-\lambda I)$ is closed and $N(T_1-\lambda I)$ is of finite dimension independent of λ.[2] Our hypothesis implies that $R(T_1-\lambda I)$ is closed for λ in Γ, but admits certain instances of $N(T_1-\lambda I)$ being infinite dimensional.

We find it convenient to employ several lemmas in the proof of the theorem. The idea for Lemma 1 came from a paper by I. C. Gokhberg

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[2] It is shown in the author's dissertation that if $R(T_1-\lambda_0 I)$ is closed and $N(T_1-\lambda_0 I)$ is finite dimensional then there exists a regular extension at λ_0. Such an extension is shown to be regular near λ_0 if and only if the dimension of $N(T_1-\lambda I)$ is constant in some neighborhood of λ_0.

and A. S. Markus [1] dealing with an operator at points of regular type.

LEMMA 1. *Let T be a regular extension near λ_0 and $\{\lambda_j\}$ be a sequence such that* $\lim_{j\to\infty}\lambda_j=\lambda_0$ *and* $\lambda_j\neq\lambda_0$ *for each j. If y is in $R(T_1-\lambda_jI)$ for every j then y is in $K(T-\lambda_0I)$.*

PROOF. Because of linearity we may assume without loss of generality that $\lambda_0=0$. By the definitions of regular extensions there exists a neighborhood of 0 such that for each λ in this neighborhood $(T-\lambda I)^{-1}$ exists as a bounded operator defined on $R(T_1-\lambda I)$. By possibly omitting a finite number of terms from $\{\lambda_j\}$ we may assume that $\{\lambda_j\}$ is contained in such a neighborhood. Thus y belongs to $R(T-\lambda_jI)$ for every j.

For each j let u_j be a solution of $(T-\lambda_jI)x=y$. Then $Tu_j=\lambda_ju_j+y$ and since T^{-1} is bounded,

$$\begin{aligned}\|u_j\| &\leqq \|T^{-1}\|\cdot\|\lambda_ju_j+y\|\\ &\leqq \|T^{-1}\|\cdot|\lambda_j|\cdot\|u_j\|+\|T^{-1}\|\cdot\|y\|.\end{aligned}$$

Thus for sufficiently large j we have

$$\|u_j\|\leqq\frac{\|T^{-1}\|\cdot\|y\|}{1-|\lambda_j|\cdot\|T^{-1}\|}\leqq 2\|T^{-1}\|\cdot\|y\|,$$

and $\{u_j\}$ is a bounded sequence.

Now $\lim_{j\to\infty}Tu_j=y$ so y belongs to $R(T)$ which is closed since it is the domain of the closed, bounded operator T^{-1}. Let x_0 be the solution of $Tx=y$. Notice that $(1/\lambda_j)(u_j-x_0)=T^{-1}u_j$ and because $\{\|T^{-1}u_j\|\}$ is bounded we have $\lim_{j\to\infty}u_j=x_0$. Assume the induction hypothesis that $x_0, x_1, \cdots, x_n$ satisfy the following conditions: $Tx_0=y$, $Tx_k=x_{k-1}$ $(k=1, 2, \cdots, n)$ and $\lim_{j\to\infty}w_{j,n}=x_n$ where

$$w_{j,n}=\left(\frac{1}{\lambda_j}\right)^n\left[u_j-\sum_{k=0}^{n-1}(\lambda_j)^kx_k\right].$$

Then $\lim_{j\to\infty}w_{j,n+1}$ exists since $\{w_{j,n}\}$ converges, T^{-1} is bounded and $Tw_{j,n+1}=w_{j,n}$. Let $x_{n+1}=\lim_{j\to\infty}w_{j,n+1}$ and observe $Tx_{n+1}=x_n$ because T is closed. Thus by induction y belongs to $K(T)$.

LEMMA 2. *If T is a regular extension near λ_0 then $N(T_1-\lambda_0I)$ is contained in $K(T-\lambda_0I)$.*

PROOF. Again assume $\lambda_0=0$ and $\{\lambda_j\}$ is a sequence such as in Lemma 1. Suppose $T_1x=0$. Then $(T_1-\lambda_jI)(x/-\lambda_j)=x$ and x belongs to $K(T)$ by Lemma 1.

LEMMA 3. *If T is a regular extension near λ, m is a non-negative integer and $x \in D((T_1 - \lambda I)^{m+1})$ then $(T - \lambda I)^{-1}(T_1 - \lambda I)^{m+1}x \in R((T_1-\lambda I)^m)$.*

PROOF. Let $\lambda=0$, $S=T^{-1}$, $x\in D(T_1^{m+1})$ and $z=T_1^m x-ST_1^{m+1}x$. Then $T_1z=T_1^{m+1}x-T_1ST_1^{m+1}x=0$ so by Lemma 2, $z\in K(T)$ which is contained in $K(T_1)$. Let v be such that $T_1^m v=z$ and observe $T_1^m(x-v) = T_1^m x-T_1^m x+ST_1^{m+1}x=ST_1^{m+1}x$.

LEMMA 4. *If T is a regular extension near λ, $S=(T-\lambda I)^{-1}$ and $y\in K(T_1-\lambda I)$ then for each non-negative integer n, $S^n y$ is defined and belongs to $K(T_1-\lambda I)$.*

PROOF. Let $\lambda=0$ and $y\in K(T_1)$. Then $S^0y=y\in K(T_1)$. Assume the induction hypothesis that $S^ny\in K(T_1)$. Since $K(T_1)$ is contained in $R(T_1)=D(S)$ we see that $S^{n+1}y$ is defined. Consider the equation $T_1^m x=S^{n+1}y$. By the induction hypothesis there is a u such that $T_1^{m+1}u=S^ny$ and then by Lemma 3 there is a v such that $T_1^m v=ST_1^{m+1}u$. Thus $S^{n+1}y=SS^ny=ST_1^{m+1}u=T_1^m v$ and since m is arbitrary we have $S^{n+1}y\in K(T_1)$.

Note that this shows $K(T-\lambda I)=K(T_1-\lambda I)$.

LEMMA 5. *Let T be a regular extension near λ_0 and $S=(T-\lambda_0 I)^{-1}$. Then y belongs to $R(T_1-\lambda I)$ for y in $K(T_1-\lambda_0 I)$ and for all λ such that $|\lambda-\lambda_0|\cdot\|S\|<1$.*

PROOF. Let $\lambda_0=0$, $y\in K(T_1)$ and λ satisfy $|\lambda|\cdot\|S\|<1$. By Lemma 4, S^ny is defined for all n and $\sum_{j=0}^{\infty}(\lambda S)^j y$ is convergent because $\|\lambda S\|<1$. Since S has a closed domain we see that $u=\sum_{j=0}^{\infty}(\lambda S)^j y \in D(S)$. Finally $x=Su$ satisfies

$$\begin{aligned}(T_1-\lambda I)x &= (T_1-\lambda I)S\sum_{j=0}^{\infty}(\lambda S)^j y\\ &= T_1Sy + T_1S\sum_{j=1}^{\infty}(\lambda S)^j y - \lambda S\sum_{j=0}^{\infty}(\lambda S)^j y\\ &= y+\sum_{j=1}^{\infty}(\lambda S)^j y-\sum_{j=0}^{\infty}(\lambda S)^{j+1}y = y.\end{aligned}$$

LEMMA 6. *$K(T_1-\lambda I)$ is independent of λ in Γ.*

PROOF. For each element y let Γ_y be the set of all $\lambda\in\Gamma$ such that $y\in K(T_1-\lambda I)$. We will show that Γ_y is both closed and open in Γ and hence either void or all of Γ since Γ is connected.

If $\lambda_0\in\Gamma\cap\overline{\Gamma}_y$ then $\lambda_0\in\Gamma_y$ by Lemma 1. Hence Γ_y is closed in Γ.

If $\lambda_0 \in \Gamma_y$ then Lemma 5 implies the existence of a neighborhood of λ_0 which is contained in Γ and such that $y \in R(T_1 - \lambda I)$ for every λ in this neighborhood. For each such λ we see that $\lambda \in \Gamma_y$ by Lemma 1. Thus Γ_y is open in Γ.

With Lemma 6, the following observation completes the proof of the theorem.

Suppose $\lambda_0 \in \Gamma$ is an accumulation point of the set of all λ such that $y \in R(T_1 - \lambda I)$. By Lemma 1, $y \in K(T_1 - \lambda_0 I)$ since $K(T_1 - \lambda_0 I) = K(T - \lambda_0 I)$ if T is a regular extension near λ_0.

References

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University of California, Berkeley and
Iowa State University