

EXTENDED CANONICAL SYSTEMS

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Let L be a (finite) alphabet containing K as a sub-alphabet; let (S) be a system, like a Post canonical system [1; 2], except that the production variables range over all strings in K (rather than all strings in L , as in Post). The symbols in L may, however, occur in the axioms and production statements of (S) . Such a system we call an *extended* canonical system; more specifically an L - K system. We say that the system is *in* the alphabet L , but *over* the alphabet K . We have found such systems to be more wieldy than the Post systems; fewer axioms are usually required, and the axioms are usually shorter.² It is easy to show the equivalence of representability in an extended canonical system to representability in a canonical system. It is well known that if K contains only 1 symbol then not every recursively enumerable set of strings in K is representable in a canonical system in the alphabet K ; only the recursive sets can be so represented. We raise the problem: if K contains only 1 symbol, is every r.e. set of strings in K representable in some *extended* canonical system *over* K ? We answer this question affirmatively.

To simplify our proof, somewhat, we shall appeal to Post's normal form theorem.³ We let K be an alphabet containing just one symbol; call this symbol "1". We shall identify a string of 1's of length n with the positive integer n . Let A be a recursively enumerable set (of positive integers). Appealing to Post's normal form theorem, there is a normal canonical system (C) in the alphabet $\{1, b\}$ and a string σ such that for every (positive) integer n , $n \in A$ iff σn is provable in (C) . We let K_2 be the alphabet $\{1, b\}$ and we let L be the 8-symbol alphabet $\{1, b, N, C, T_0, P, Q, -\}$. We shall construct an L - K system in which A is represented. Along the way, we will have to represent certain *relations* of numbers (strings in K). For any L - K system (S) , any string π in L , and any relation $R(x_1, \dots, x_n)$ of strings in K , the string π is said to *represent* R iff the following condition holds:

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² Extended canonical systems come very close to the elementary formal systems defined in [3].

³ It is not difficult to modify our proof so as to circumvent the normal form theorem. The main point of using the normal form theorem is to avoid the heavy meta-mathematical notation for general productions.

for any strings X_1, \dots, X_n in K , $R(X_1, \dots, X_n)$ holds iff the string $\pi X_1 - X_2 - \dots - X_n$ is provable in (S) . [For a set W of strings in K , π represents W in $(S) =_d$ for every string X in K , $X \in W$ iff πX is provable in (S)].

We assign Gödel numbers to all strings in K_2 as follows. We define $g(1)=1$, $g(b)=2$, and for any string $d_n d_{n-1} \dots d_1 d_0$ (where each d_i is either 1 or b) we define $g(d_n d_{n-1} \dots d_1 d_0)$ to be the number $g(d_0) + 2 \cdot g(d_1) + 4 \cdot g(d_2) + \dots + 2^n \cdot g(d_n)$. The Gödel correspondence g is 1-1 from the set of all strings in K_2 onto all the positive integers. For any positive integer i we let E_i be that string in K_2 whose Gödel number is i . By $E_i E_j$ we mean E_i followed by E_j , and we let $i * j$ be the Gödel number of $E_i E_j$. The function $x * y$ obeys the following conditions:

- (1) $x * 1 = 2x + 1$,
- (2) $x * 2 = 2x + 2$,
- (3) $(x * y) * z = x * (y * z)$.

Conversely, these 3 conditions uniquely determine the function $x * y$. We can thus represent the relation $x * y = z$ in the following L - K system. [We are using x, y, z, u, v, w , as production variables. And, of course, these variables range over all strings in K .]

Axioms. N1

Productions.

$$Nx \rightarrow Nx1,$$

$$Nx \rightarrow Cx - 1 - xx1,$$

$$Nx \rightarrow Cx - 11 - xx11,$$

$$Cx - y - w, \quad Cy - z - v, \quad Cw - z - u \rightarrow Cx - v - u.$$

In the above L - K system—call it (S_1) —the symbol “ N ” represents the set of all positive integers (i.e. strings in K) and “ C ” represents the relation $x * y = z$.

We let T be the set of all strings provable in the normal canonical system (C) , and we let T_0 be the corresponding set of Gödel numbers. We now wish to extend (S_1) to an L - K system (S_2) in which T_0 is represented. We add to (S_1) the axiom $T_0 n$, where E_n is the one axiom of (C) . And for each normal production $E_i x \rightarrow x E_j$ of (C) , we add to (S_1) the production $Ci - x - y, Cx - j - z, T_0 y \rightarrow T_0 z$. In this system (S_2) , “ T_0 ” represents the set of all numbers n such that E_n is provable in (C) —i.e. “ T_0 ” represents T_0 .

The set A is represented by some string E_d in (C) . Thus A con-

sists of the set of all n such that $E_d n \in T$. The Gödel number of E_d is, of course, d and the Gödel number of n (i.e. of a string of 1's of length n) is $2^n - 1$. Thus A consists of all n such that $d * (2^n - 1) \in T_0$. We add to (S_2) :

Axiom. $P1 - 11$.

Production. $Px - y \rightarrow Px1 - yy$.

P represents the set of all ordered pairs (i, j) such that $2^i = j$.

Then we add:

Production. $Px - y1, Cd - y - z, T_0 z \rightarrow Qx$.

In this system (S) , " Q " represents A .

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THE AXIOM FOR CONNECTED SETS

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1. Introduction. The basic motivation for this study was a desire to find a genuinely unified postulational principle which incorporated both the Axiom of Choice and the "axiom for sets," which latter means an appropriate analogue of the *Aussonderungssaxiom* to provide for the existence of sets. The possibility of thus uniting these two axiomatic principles has become especially interesting since adding the Axiom of Choice has been shown to be not only safe,¹ but necessary as well.² In particular, it was further hoped and expected that such a principle, when found, could be expressed naturally as a membership-equivalence statement—that is, essentially of the form

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¹ Gödel showed [1] that for certain systems of set theory, adding the Axiom of Choice does not bring inconsistency.

² Mendelson showed [2] that the Axiom of Choice is independent of the other, usual axioms for set theory, hence is indispensable for developments which employ it essentially.