BIBLIOGRAPHY

1. J. Lillo, *Continuous matrices and the stability theory of differential systems*, Math. Z. vol. 73 (1960) pp. 45–58.

2. ———, *Perturbations of nonlinear systems*, Acta Math. vol. 103 (1960) pp. 123–138.

3. ———, *Linear differential equations with almost periodic coefficients*, Amer. J. Math. vol. 81 (1959) pp. 37–45.

4. L. Markus, *Continuous matrices and the stability of differential systems*, Math. Z. vol. 62 (1955) pp. 310–319.

5. W. T. Reid, *Remarks on a matrix transformation for linear differential equations*, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 708–712.

RIAS

METRIC TERNARY DISTRIBUTIVE SEMI-LATTICES

S. P. AVANN

In this paper we show that the ternary operation of a metric ternary distributive semi-lattice, a generalization of the ternary Boolean algebra of Grau [2], uniquely minimizes ternary distance. This generalizes a result of Birkhoff and Kiss [1, Corollary 1, p. 749]. We show, conversely, that in a metric space unique minimizing of ternary distance determines a ternary operation with respect to which the space is a ternary distributive semi-lattice. Particularly, a lattice whose graph satisfies the unique minimal ternary distance condition and certain finiteness conditions must be distributive. This answers a question proposed by Birkhoff and Kiss [1, p. 750].

1. Definitions and postulates.

We state our results at the close of this section.

A ternary distributive semi-lattice, hereinafter abbreviated TDSL, is a set of $\mathfrak{I}$ elements closed with respect to a ternary operation (a, b, c) satisfying the following identities.

(T1) $(a, a, b)=a$.

(T2) (a, b, c) is invariant under all 6 permutations.

(T3) $(a, (b, c, d), e)=((a, b, e), c, (a, d, e))$.

REMARK. The term, introduced by the author (Abstract 86, Bull. Amer. Math. Soc. vol. 54 (1948) p. 79), is a natural one in view of Lemma 3. If in Lemma 3 there exists $a'\in\mathfrak{I}$ satisfying

Received by the editors April 11, 1960 and, in revised form, July 18, 1960.

(T4) $(a, b, a') = b$ for all $b \in \mathfrak{J}$

then $\mathcal{P}(a, \mathfrak{J})$ is a distributive lattice with a and a' as zero and unit elements. If also $\mathfrak{J}$ satisfies:

(T5) For each $a \in \mathfrak{J}$ there exists a complement $a' \in \mathfrak{J}$ satisfying (T4), then $\mathfrak{J}$ becomes the Ternary Boolean Algebra of Grau [**2**] and $\mathcal{P}(a, T)$ is a Boolean Algebra for each $a \in \mathfrak{J}$.

By a suitable permutation of the letters in (T3) Sholander in [**4**, p. 801] was able to replace (T2) and (T3) by a single postulate (N). His (M) is (T1).

We remark here that by virtue of (T2), (T3) can be written and applied with many variations; particularly, the solo element in the right member can be b or d.

In a metric space $\mathfrak{M}$ we denote distance by bc and introduce ternary distance $[x; b, c, d] = xb + xc + xd$.

We shall be concerned with an undirected graph $\mathcal{G}$ with no loops, i.e., the graph of a symmetric anti-reflexive binary relation R on a set of elements: aRa is false for all $a \in \mathcal{G}$ and aRb iff bRa. Two elements b and c are vertices of an edge iff bRc. Moreover, when $\mathcal{G}$ is connected, it is a metric space with respect to distance defined: $bb = 0$; $bc = 1$ iff bRc; $bc = n$ iff $bRb_1R \cdots Rb_n = c$ is a minimal such sequence. An even graph is one with no odd-sided polygons $b_1Rb_2R \cdots Rb_{2n+1}Rb_1$.

The graph $\mathcal{G}(\mathcal{P})$ of a partially ordered set $\mathcal{P}$ is defined by: bRc iff $b < c$ or $b > c$ ($<$: is covered by).

We shall deal with the following two minimal ternary distance postulates in a metric space $\mathfrak{M}$ and a corresponding ternary operation for each.

(U) For each (unordered) triple $b, c, d \in \mathfrak{M}$ there exists a unique $t \equiv [b, c, d] \in \mathfrak{M}$ such that $[t; b, c, d] = (bc + cd + db)/2$.

(V) For each triple $b, c, d \in \mathfrak{M}$ there exists a unique $s \equiv [b, c, d] \in \mathfrak{M}$ such that $[s; b, c, d] < [x; b, c, d]$ for all $x \in \mathfrak{M}$, $x \neq s$.

By virtue of Lemma 1 we shall see that (U) implies (V) in $\mathfrak{M}$.

Ternary betweenness relations and notation are defined as follows:

(TB) In a TDSL $\mathfrak{J}$, $(bxc) \leftrightarrow (b, x, c) = x$.

(MB) In a metric space $\mathfrak{M}$, $bxc \leftrightarrow bx + xc = bc$.

(VB) In a graph $\mathcal{G}$ satisfying (V) or (U), $[bxc] \leftrightarrow [b, x, c] = x$.

Finiteness conditions in terms of convex sets are defined as follows:

(TF) In a TDSL $\mathfrak{J}$, $\{x \in \mathfrak{J} \mid (bxc)\}$ is finite for all $b, c \in \mathfrak{J}$.

(MF) In a metric space $\mathfrak{M}$, $\{x \in \mathfrak{M} \mid bxc\}$ is finite for all $b, c \in \mathfrak{M}$.

(VF) In a graph $\mathcal{G}$ satisfying (V) or (U), $\{x \in \mathcal{G} \mid [bxc]\}$ is finite for all $b, c \in \mathcal{G}$.

When (TF) holds we define the graph $\mathcal{G}(\mathfrak{J})$ of a TDSL $\mathfrak{J}$ as follows: bRc iff $b \neq c$ and $(b, x, c) = b$ or c for all $x \in \mathfrak{J}$. $\mathcal{G}(\mathfrak{J})$ will be connected, as shown in Lemmas 4 and 7, and therefore metrizable in the manner described above.

We now summarize our results.

THEOREM 1. *If $\mathfrak{I}$ is simultaneously a TDSL and a metric space in which* (TB) *and* (MB) *are equivalent*: $(bxc) \leftrightarrow bxc$, *then* (U) *is satisfied* (*and also* (V)).

THEOREM 2. *If a TDSL $\mathfrak{I}$ satisfies* (TF), *then the metric space $\mathcal{G}(\mathfrak{I})$, as defined and metrized above, satisfies* (U). *Moreover* (TB) *and* (MB) *are equivalent.*

THEOREM 3. *A metric space $\mathfrak{M}$ satisfying* (U) *is a TDSL with respect to the ternary operation* $[b, c, d]$. *Moreover* (MB) *is equivalent to* (VB) (*which is* (TB)).

We define a unique ternary distance graph $\mathcal{G}$, hereinafter called a UTD graph, as one satisfying (MF) and (V).

THEOREM 4. *A UTD graph satisfies* (U) *and is a TDSL with respect to the ternary operation* $[b, c, d]$. *Moreover* (MB) *and* (VB) *are equivalent.*

THEOREM 5. *If every $a \in \mathfrak{L}$, a lattice with zero element z, is of finite dimension, and if the graph $\mathcal{G}(\mathfrak{L})$ satisfies* (MF) *and* (V), *then $\mathfrak{L}$ is distributive.*

2. **Ternary distributive semi-lattices.** In this section we consider a TDSL which is a metric space and prove Theorems 1 and 2.

LEMMA 1. *In any metric space $\mathfrak{M}$*

(MT1) $[x; b, c, d] \geqq (bc+cd+db)/2$,

(MT2) $[x; b, c, d] = (bc+cd+db)/2 \leftrightarrow bxc \cdot cxd \cdot dxb$.

PROOF. (MT1) follows from taking one-half the sum of the inequalities $bx+xc \geqq bc$, $cx+xd \geqq cd$, $dx+xb \geqq db$. Clearly equality holds simultaneously in all three iff equality holds in (MT1).

LEMMA 2. *In a TDSL $\mathfrak{I}$ $(btc) \cdot (ctd) \cdot (dtb)$ is satisfied uniquely by $t = (b, c, d)$, where $\cdot$ denotes logical conjunction.*

This follows easily from (T1-2-3). See [**3**, 8.4 and 8.13].

PROOF OF THEOREM 1. Since (TB)$\leftrightarrow$(MB), by Lemma 2 we have $btc \cdot ctd \cdot dtb$ holding uniquely for $t = (b, c, d)$. Whence by (MT2) and (MT1) resp. $[t; b, c, d] = (bc+cd+db)/2 < [x; b, c, d]$ for all $x \neq t$.

LEMMA 3. *For each $a \in \mathfrak{I}$, a TDSL, the elements of $\mathfrak{I}$ constitute a distributive semi-lattice $\mathcal{P}(a, \mathfrak{I})$, closed with respect to symmetric join of* *meets of triples (called by Sholander a median semi-lattice) as follows:*

(1) *The inclusion relation is given by* $b \subseteq_a c$ *(and* $c \supseteq_a b$*)* $\leftrightarrow (a, b, c) = b$.

(2) *The zero element is* a.

(3) $\mathcal{P}(a, \mathfrak{J})$ *is closed with respect to meet given by* $b \cap_a c = (b, a, c)$.

(4) *Existence of common upper bound* $b \subseteq_a m$ *and* $c \subseteq_a m$, *implies the join exists and is given by* $b \cup_a c = (b, m, c)$.

(5) *Distributivity: existence of* $b \cup_a c$ *implies* $d \cap_a (b \cup_a c) = (d \cap_a b) \cup_a (d \cap_a c)$.

(6) *For all triples* b, c, d *there exists* $(b \cap_a c) \cup_a (c \cap_a d) \cup_a (d \cap_a b)$, *which is* (b, c, d).

The proof is a routine application of the postulates and is done in [**5**, pp. 809–810].

LEMMA 4. *Every principal ideal of* $\mathcal{P}(a, \mathfrak{J})$, *namely* $\mathcal{P}(a, m) = \{x \mid (axm)\}$, *is a distributive lattice, which is finite if* (TF) *is satisfied.*

PROOF. The lemma follows from (4) of Lemma 3 and the fact that one distributive law implies the other.

LEMMA 5. *In a TDSL* $\mathfrak{J}$ $(abc) \cdot (acd) \leftrightarrow (abd) \cdot (bcd)$.

We prove this known result to illustrate applications of the postulates. If $(abc) \cdot (acd)$, then $(a, b, d) = (a, (a, b, c), d) = ((a, a, d), b, (a, c, d)) = (a, b, c) = b$ yielding (abd). Also $(b, c, d) = ((a, b, c), c, d) = ((a, c, d), b, (c, c, d)) = (c, b, c) = c$ so that (bcd) subsists. The converse holds by symmetry.

LEMMA 6. *In* $\mathcal{P}(a, \mathfrak{J})$, b *is covered by* $c \neq b$: $b <_a c$ $(c >_a b)$ *iff* $(a, b, c) = b$ *and* $(b, x, c) = b$ *or* c *for all* $x \in \mathfrak{J}$.

PROOF. Let $b <_a c$. Then $(b, a, c) = (a, b, c) = b$ and (abc). For arbitrary $x \in \mathfrak{J}$ let $(b, x, c) = d$. Then also (bdc) by Lemma 2. Applying Lemma 5 with roles of c and d interchanged, we obtain $(abd) \cdot (adc)$. By Lemma 3 $a \subseteq_a b \subseteq_a d \subseteq_a c$, and the hypothesis requires $d = b$ or $d = c$. Conversely, let $(b, x, c) = b$ or c for all x and $(b, a, c) = b$. Then immediately $b \subseteq_a c$. Assume $a \subseteq_a b \subseteq_a x \subseteq_a c$ so that $(abx) \cdot (axc) \cdot (abc)$. By Lemma 5 (bxc) so that $x = (b, x, c)$, which must be b or c as desired.

LEMMA 7. *In a TDSL* $\mathfrak{J}$ *satisfying* (TF), bRc *in* $\mathcal{G}(\mathfrak{J})$ *iff* bR_ac *in* $\mathcal{G}(\mathcal{P}(a, \mathfrak{J}))$, *where* R_a *is* $\lessgtr_a$. *Thus* $\mathcal{G}(\mathfrak{J})$ *and* $\mathcal{G}(\mathcal{P}(a, \mathfrak{J}))$ *are isometric.*

PROOF. In $\mathcal{G}(\mathfrak{J})$ bRc iff $(b, x, c) = b$ or c for all x including $(b, a, c) = b$ or c. Hence by Lemma 6 bRc iff $b \lessgtr_a c$ in $\mathcal{P}(a, \mathfrak{J})$ iff bR_ac in $\mathcal{G}(\mathcal{P}(a, \mathfrak{J}))$.

LEMMA 8. *In a TDSL* $\mathfrak{J}$ *satisfying* (TF), (TB) *and* (MB) *are equivalent.*

PROOF. Given (abc). Then $(a, b, c)=b$ and $b\subseteq_a c$ in the principal ideal $\mathcal{P}(a, c)$. The latter is a finite distributive lattice by Lemma 4 and satisfies the Jordan-Dedekind chain condition. Therefore a chain $a<_a a_1<_a \cdots <_a a_m=b\in\mathcal{P}(a, c)$ exists and minimizes a sequence $aR_a x_1 R_a \cdots R_a b$, where R_a is $\lessgtr_a$. The corresponding sequence of $\mathcal{G}(\mathfrak{I})$ of Lemma 7: $aRa_1R \cdots Ra_m=b$ is thus minimal so that $ab=\delta_a[b]$, the dimension of b in $\mathcal{P}(a, \mathfrak{I})$. Similarly $ac=\delta_a[c]$. Again, $b<_a b_1<_a \cdots <_a b_n=c$ minimizes sequences $bR_a \cdots R_a c$ and by virtue of Lemma 7 yields a corresponding minimal sequence $bRb_1R \cdots Rb_n=c$ of $\mathcal{G}(\mathfrak{I})$ of length $bc=n$. The total chain $a<_a a_1<_a \cdots <_a a_m=b<_a b_1<_a \cdots <_a b_n=c$, again in view of the Jordan-Dedekind chain condition in $\mathcal{P}(a, c)$, yields a minimal chain $aRa_1R \cdots Ra_m=bRb_1R \cdots Rb_n=c$. Hence $ac=\delta_a[c]=\delta_a[b]+bc=ab+bc$, yielding abc. Conversely suppose abc holds. Let $d=(a, b, c)$. By Lemma 2 $(adb)\cdot(bdc)\cdot(cda)$. By the proof just completed $adb\cdot bdc\cdot cda$. Hence $0 = (ad + db - ab)/2 + (bd + dc - bc)/2 - (cd + da - ca)/2 = bd-(ab+bc-ca)/2=bd-(ac-ca)/2=bd$. Thus $b=d$, $(a, b, c)=b$, and (abc).

PROOF OF THEOREM 2. Lemma 8 completes the hypothesis of Theorem 1.

3. Unique ternary distance graphs.

We prove Theorems 3, 4 and 5 in this section.

LEMMA 9. *In a metric space satisfying* (U), (VB) *and* (MB) *are equivalent.*

PROOF. $[bcd]$ iff $c = [b, c, d]$ iff $[c;\ b, c, d] = bc + cc + cd = (bc+cd+db)/2$ iff $bc+cd=bd$ iff bcd.

LEMMA 10. (Condition (D) of Sholander [4, p. 804]). *For each unordered triple $b, c, d\in\mathfrak{M}$, a metric space satisfying* (U), *there exists a unique $s\in\mathfrak{M}$ such that $bsc\cdot csd\cdot dsb$, namely $s=[b, c, d]$.*

PROOF. By (U) there exists unique $s=[b, c, d]$ such that $[s; b, c, d]=(bc+cd+db)/2$. We apply Lemma 1. By (MT2) $bsc\cdot csd\cdot dsb$, and for $x\neq s$ $[x; b, c, d]>(bc+cd+db)/2$ so that at least one of bxc, cxd, dxb fails.

LEMMA 11. *In any metric space $abc\cdot acd\leftrightarrow abd\cdot bcd$.*

This is an elementary property of metric spaces.

PROOF OF THEOREM 3. The metric betweenness relation bcd satisfies the set of conditions $\Sigma_1(D, B_1, F)$ of Sholander [4, pp. 803–805]: (D) by Lemma 10; (B_1) $aba\rightarrow a=b$, trivially; and (F) $abc\cdot acd$

$\rightarrow dba$ ($\leftrightarrow abd$) by Lemma 11. By Lemma 9 the equivalent betweenness relation $[bcd]$ also satisfies Σ_1. Sholander showed in [4, 4.10] that the corresponding ternary operation $[b, c, d]$ satisfies his conditions (M) and (N). The latter, he showed in [3, 8.3], are equivalent to (T1-2-3).

COROLLARY TO THEOREM 3. *If a metric space* $\mathfrak{M}$ *satisfies* (U) *and for some pair* $a, a' \in \mathfrak{M}$ axa' *for all* $x \in \mathfrak{M}$, *then* $\mathcal{P}(a, \mathfrak{J}) = \mathcal{P}(a, a')$ *is a distributive lattice with a and a' as zero and unit elements.*

PROOF. By Theorem 3 $\mathfrak{M}$ becomes a TDSL $\mathfrak{J}$ under $[b, c, d]$ with $[axa']$ for all $x \in \mathfrak{J}$. I.e., $a \subseteq_a x \subseteq_a a'$ for all x. Lemma 4 completes the proof.

LEMMA 12. *A necessary and sufficient condition that a connected graph* $\mathcal{G}$ *be even is that bRc implies* $bx - cx = \pm 1$. *Furthermore, a UTD graph is even.*

PROOF. Given $\mathcal{G}$ is even and suppose bRc. Then $1 = bc \geqq bx - cx \geqq -bc = -1$. But $bx \neq cx$ since $bx - cx + 1 \equiv bx + cx + bc \equiv 0 \pmod 2$. Hence $bx - cx = \pm 1$. Conversely suppose $\mathcal{G}$ is not even. Two adjacent vertices b, c and the opposite vertex x of a smallest odd-sided polygon give $bc = 1$ and $bx = cx$. Moreover $[c; x, b, c] = cx + 1 = bx + 1 = [b; x, b, c] \leqq xy + by + cy = [y; x, b, c]$ for $b \neq y \neq c$. Hence b and c (and possibly y also) are tied for minimal ternary distance from x, b, c so that $\mathcal{G}$ is not a UTD graph. We have thus proved the contrapositives of the converse and the second statement.

We may note at this point that in a UTD graph the ternary operation $[b, c, d]$ satisfies (T2) trivially by symmetry. It also satisfies (T1). For if $x \neq a$, $[x; a, a, b] = ax + (ax + bx) > aa + aa + ab = [a; a, a, b]$ so that $a = [a, a, b]$. We shall circumvent a direct proof of (T3), which would be tedious.

LEMMA 13. *In a UTD graph* (MB) *and* (VB) *are equivalent.*

PROOF. First suppose abc. If $x \neq b$, $[x; a, b, c] = (ax + cx) + bx > ac = ab + bc = [b; a, b, c]$. Hence $b = [a, b, c]$ and $[abc]$ subsists. Conversely, suppose $[abc]$. We shall prove by induction on $n = bc$ that abc follows. We note that abc holds trivially for $n = 0$. When $n = 1$, $ab = ac \pm 1$ by Lemma 12. But $ab = ac + 1 = ac + bc$ yields acb and $[acb]$ by the first part of this proof and leads to the contradiction $[a, b, c] = [a, c, b] = c \neq b = [a, b, c]$. Thus $ac = ab + bc$ as desired. Assume $[a, b, c] = b$ implies abc whenever $n \leqq k$. Consider $[a, b, c] = b$ with $n = bc = k + 1$. Let bRb_0 with b_0 on minimal $b - c$ chain: $bb_0 = 1$ and $bb_0 + b_0c = bc$. Since $b = [a, b, c]$, $ab + bc = [b; a, b, c] < [b_0; a, b, c]$

$=ab_0+(bb_0+b_0c)=ab_0+bc$. Thus $ab<ab_0$ and $ab_0=ab+1$ by Lemma 12. Now $[b;\ a,\ b_0,\ c]=ab+1+bc>ab+bc=ab+1+b_0c=ab_0+b_0c$ $=[b_0;\ a,\ b_0,\ c]$. Also for $x\neq b$ we apply hypothesis and Lemma 12 to obtain $[x;\ a,\ b_0,\ c]=[x;\ a,\ b,\ c]+(b_0x-bx)\geqq 1+[b;\ a,\ b,\ c]+(\pm 1)$ $\geqq [b;\ a,\ b,\ c]=ab+bc=ab_0+b_0c=[b_0;\ a,\ b_0,\ c]$. Uniqueness of minimality in (V) requires that $b_0=[a,\ b_0,\ c]$ or $[ab_0c]$. But $b_0c=bc-1=k$. By the induction hypothesis ab_0c subsists. Hence $ac=ab_0+b_0c$ $=ab+bc$ yielding abc. The induction is complete.

Proof of Theorem 4. Let $s=[b,\ c,\ d]$. Then for $x\neq s[s;\ b,\ s,\ c]$ $=(bs+cs+ds)-ds<bx+cx+(dx-ds)\leqq bx+cx+sx=[x;\ b,\ s,\ c]$. Thus $s=[b,\ s,\ c]$ and $[bsc]$ subsists. Similarly $[csd]$ and $[dsb]$. Then $bsc\cdot csd\cdot dsb$ by Lemma 13. By Lemma 1 and (V) we have $[s;\ b,\ c,\ d]$ $=(bc+cd+db)/2<[x;\ b,\ c,\ d]$ for all $x\neq s$. This is (U). Hence Theorem 4 now follows from Theorem 3.

Lemma 14. *In any lattice or semi-lattice if aRbRcRdRa, where R is* $\lessgtr$, *alternate R's are opposite directional covering.*

This follows by definition of covering and uniqueness of join and meet when they exist.

Lemma 15. *In a lattice* $\mathfrak{L}$, *for which* $\mathfrak{G}(\mathfrak{L})$ *is a UTD graph with respect to the ternary operation determined by the metric,* $b\lessgtr c$ *in* $\mathfrak{L}$ *iff* $b\neq c$ *and* $[b,\ x,\ c]=b$ *or* c *for all* $x\in\mathfrak{G}(\mathfrak{L})$.

Proof. If $b\lessgtr c$ in $\mathfrak{L}$, then by Lemma 12 $\mathfrak{G}(\mathfrak{L})$ is even and $bx-cx$ $=\pm 1=\pm bc$ in $\mathfrak{G}(\mathfrak{L})$. Thus $[bcx]$ or $[cbx]$, i.e., $[b,\ x,\ c]=c$ or b. If $b\not\lessgtr c$, then $b=c$ or there exists $r\in\mathfrak{G}(\mathfrak{L})$ with $b,\ r,\ c$ all distinct such that $[brc]$ or $[b,\ r,\ c]=r\neq b$ or c.

Proof of Theorem 5. Finite dimensionality of the elements of $\mathfrak{L}$ makes $\mathfrak{G}(\mathfrak{L})$ well defined and connected through z, so that it is a UTD graph. Hence by Theorems 3 and 4 $\mathfrak{G}(\mathfrak{L})$ is a TDSL with respect to the operation $[a,\ b,\ c]$, and (MB) is equivalent to (VB) (which is (TB)). Moreover all the lemmas are valid and applicable. By Lemma 3 $\mathcal{P}(z,\ \mathfrak{G}(\mathfrak{L}))$ is a distributive semi-lattice with the same zero element z of $\mathfrak{L}$. We shall show that $\mathcal{P}(z,\ \mathfrak{G}(\mathfrak{L}))$ is isomorphic to $\mathfrak{L}$ under the identity correspondence $c\leftrightarrow c$. Combining the results of Lemmas 6 and 15 $b\lessgtr_z c$ in $\mathcal{P}(z,\ \mathfrak{G}(\mathfrak{L}))$ iff $b\lessgtr c$ in $\mathfrak{L}$. Accordingly, it will be sufficient to show that

(S) $b<_z c$ in $\mathcal{P}(z,\ \mathfrak{G}(\mathfrak{L}))$ implies $b<c$ in $\mathfrak{L}$.

We employ an induction on $n=zc$, the distance from z to c in $\mathfrak{G}(\mathfrak{L})$, and note (S) is trivially true for $n=1$: $z<_z c$ in $\mathcal{P}(z,\ \mathfrak{G}(\mathfrak{L}))$ iff $z<c$ in $\mathfrak{L}$. Assume (S) holds for $n\leqq k$. Now consider $b<_z c$ with $zb=k$ and zc $=k+1$ in $\mathfrak{G}(\mathfrak{L})$, and assume that $b>c$ in $\mathfrak{L}$. From all necessarily finite descending chains $b > c > \cdots$ in $\mathfrak{L}$ select one with an earliest agreement of $>$ and $>_z$: $b = c_0 <_z c = c_1 <_z \cdots <_z c_{r-1} <_z c_r >_z c_{r+1} \lessgtr_z \cdots$. By Lemma 4 the ideal $\mathcal{P}(z, c_r)$ of $\mathcal{P}(z, \mathcal{J}(\mathfrak{L}))$ is a distributive lattice. Hence by lower semi-modularity $c_{r-1} >_z d = c_{r-1} \cap_z c_{r+1} <_z c_{r+1}$. On the other hand $c_{r-1} > c_r > c_{r+1} < d < c_{r-1}$, where the direction of the last two coverings are required by Lemma 14. If $r = 1$, the induction hypothesis requires $b > d < c_2$ contradicting $c_2 < d < b$. If $r > 1$, then $c_{r-1} > d$ contradicts the minimality of r. Hence our assumption $b > c$ is false, and the induction on $n = zc$ for validity of (S) is complete. Therefore $\mathcal{P}(z, \mathcal{J}(\mathfrak{L}))$ is isomorphic to $\mathfrak{L}$ and is a distributive lattice (rather than a semi-lattice). Thus $\mathfrak{L}$ itself is distributive.

References

1. G. Birkhoff and S. A. Kiss, *A ternary operation in distributive lattices*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 749–752.

2. A. A. Grau, *Ternary Boolean algebra*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 567–572.

3. Marlow Sholander, *Trees, lattices, order, and betweenness*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 369–381.

4. ———, *Medians and betweenness*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 801–807.

5. ———, *Medians, lattices, and trees*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 808–812.

University of Washington