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A THEOREM ON OVERCONVERGENCE

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The conjecture announced by A. J. Macintyre [2; 3] is equivalent to the theorem stated and proved below.

Theorem. *Let D be an open domain containing the origin and let $f(z)$ be a function regular in D with the expansion $f(z)=\sum_0^\infty c_n z^n$. Let D_1 be a bounded closed domain contained in D. Then there exists a positive number $\lambda_0=\lambda_0(D, D_1)$ such that if $c_n=0$ for a sequence of intervals $n_k \leqq n \leqq \lambda n_k$ with $\lambda > \lambda_0$, then the subsequence of partial sums $s_{n_k}=\sum_0^{n_k} c_n z^n$ converges uniformly to $f(z)$ in D_1.*

Proof. Let CD and CD_1 denote the complements of D and D_1 respectively and let h_i, $i=1, 2, \cdots$, be the components of CD_1. The components can be considered as disjoint and there exists only one unbounded component. The one unbounded component will be denoted as h_1.

One can assert that there exists only a finite number of components h_i such that

$$h_i \cap CD \neq \emptyset, \tag{1}$$

where $\emptyset$ is the empty set. This assertion is proved as follows. Assume that there exists an infinite number of components h_i, $i \geqq 2$, such that (1) is valid. A bounded sequence of points a_i can be formed where $a_i \in h_i \cap CD$, $i \geqq 2$. Every a_i is an element of CD and hence the dis-

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tance d from D_1 is at least $\delta > 0$. The limit point a of the sequence then must be such that $d(a, D_1) \geqq \delta > 0$. Thus a is an element of CD_1 and all points z in $|z-a| < \delta$ must be in the same component.

Let the finite number of components be enumerated as h_i, $i=1, 2, \cdots, N$. Considering now

$$(2) \qquad D_2 = D_1 + \bigcup_{i=N+1}^{\infty} h_i,$$

then D_2 is a bounded closed domain and $D_2 \subset D_1$. Since $\sum_{i=N+1}^{\infty} h_i$ is bounded and D_1 is bounded by hypothesis, D_2 is bounded. Also, since $h_i \cap CD = \emptyset$, $i \geqq N+1$, then $h_i \subset D$ and D_2 is contained in D. To prove that D_2 is closed note that its complement is $\sum_{i=1}^{N} h_i$ and is open.

Now $N-1$ polygonal arcs $L_1, L_2, \cdots, L_{N-1}$ can be chosen such that $D_2 - \sum_1^{N-1} L_k$ is simply connected. Also $N-1$ other polygon arcs $L_1', L_2', \cdots, L_{N-1}'$ can be so chosen that $L_k \cap L_j' = \emptyset$ and $D_2 - \sum_1^{N-1} L_k'$ is simply connected. Consider now the open circle $C(s, R)$ or $|z-s| < R$ and let $S(L, R) = \bigcup_{s \in L} C(s, R)$. Thus $S(L, R)$ is a strip enclosing the polygonal arc L. For R sufficiently small,

$$(3) \qquad S(L_k, R) \cap S(L_j', R) = \emptyset.$$

Hence for R sufficiently small two closed simply connected domains can be defined, $D_3 = D_2 - \bigcup_1^{N-1} S(L_k, R)$ and $D_3' = D_2 - \bigcup_1^{N-1} S(L_j', R)$ such that $D_3 + D_3' = D_2$. This follows from

$$\begin{aligned} D_3 + D_3' &= D_2 - \left\{ \bigcup_{k=1}^{N-1} S(L_k, R) \right\} \cap \left\{ \bigcup_{j=1}^{N-1} S(L_j', R) \right\} \\ &= D_2 \end{aligned}$$

by (3).

The proof of the theorem follows. An open bounded simply connected domain $\Delta = \Delta(D, D_3)$ can now be defined such that $D_3 \subset \Delta$, $\{|z| < r\} \subset \Delta$, $\bar{\Delta} \subset D$ where r is the radius of convergence of $f(z) = \sum_0^{\infty} c_n z^n$ and $\bar{\Delta}$ is the closure of Δ. From the Nevanlinna two-constant theorem, if $F(z)$ is regular in Δ

$$M(\Delta) = \operatorname*{l.u.b.}_{z \in \Delta} |F(z)|, \qquad M(d) = \operatorname*{l.u.b.}_{|z| < r/2} |F(z)|,$$

then [1]

$$(4) \qquad M(D_3) = \operatorname*{l.u.b.}_{z \in D_3} |F(z)| \leqq \{M(\Delta)\}^{\theta} \{M(d)\}^{1-\theta}$$

where $\theta > 0$ depends on D_3 and Δ. Using the majorization of r_{n_k}, where $r_{n_k} = f(z) - s_{n_k}$, $r_{n_k} = \sum_{n_k}^{\infty} c_n z^n$, if n_k is large we get $\text{l.u.b.}_{|z| < r/2} |r_{n_k}| <$

$H(3/4)^{\lambda n_k}$ and $\text{l.u.b.}_{z\in\Delta}|r_{n_k}| < H_1^{n_k}$ where H and H_1 are two constants depending only on Δ. Thus by (4),

$$\underset{z\in D_3}{\text{l.u.b.}} |r_{n_k}| \leqq H^{1-\theta}\{H_1^{\theta}(3/4)^{\lambda(1-\theta)}\}^{n_k}.$$

Thus if $\lambda > \lambda_0(\Delta, D_3)$ there is overconvergence in D_3. Similarly there is overconvergence in D_3' if $\lambda > \lambda_0(\Delta, D_3')$. Now since $D_3 + D_3' = D_2 \supset D_1$ the theorem is proved.

REMARK. By the same method similar results are proved for the series of Dirichlet and for the integral of Laplace.

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