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ON BOUNDED FUNCTIONS WITH ALMOST PERIODIC DIFFERENCES

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The aim of this paper is to generalize to groups the well-known Bohl-Bohr theorem, which states that if the indefinite integral, $F(x) = \int_0^x f(t)dt$, of an almost periodic function $f(x)$, is bounded, then it is almost periodic (see [1] or [2]).

No expression of the form $\int_0^x f(t)dt$ is available in groups, but observing that $F(a+x) - F(x) = \int_x^{x+a} f(t)dt$ is easily proved to be almost periodic, whatever be the constant a, we are led to the following

THEOREM. *Let G be a multiplicative group and let the left differences $F(ax) - F(x)$ be right almost periodic for every $a \in G$, where F is a given complex-valued function on G. If $F(x)$ is bounded then it is right almost periodic.*

We recall that a real or complex function $\phi(x)$ is right almost periodic if, from every sequence (c_n) we can extract a subsequence (b_n) for which the functions $\phi(xb_n)$ converge uniformly in G. In that case to every $\epsilon > 0$ there corresponds a finite number of elements of G, say $s_1, \cdots, s_k$, such that to every $t \in G$ we can associate an integer $i \leq k$ for which

$$|\phi(xt) - \phi(xs_i)| < \epsilon, \qquad \text{whatever be } x \in G.$$

PROOF OF THE THEOREM. It is sufficient to consider the case of a real function. Suppose that $F(x)$ is not right almost periodic. Then there exists an $\alpha > 0$ and a sequence (c_n), such that, in every subse-

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quence $(b_n)\subset(c_n)$ we can find b_p, b_q for which $\sup|F(xb_p)-F(xb_q)|>\alpha$. We can even suppress the modulus sign by exchanging if necessary b_p and b_q. In other words there exist $t\in G$, b_p, b_q such that

$$F(tb_q^{-1}b_p) - F(t) > \alpha. \tag{1}$$

We shall prove that if $F(a_1x_1)-F(x_1)>\beta$, we can find x_2, $a_2\in G$ such that $F(a_2x_2)-F(x_2)>\beta+\alpha$. This will show that $F(x)$ is unbounded, against the hypothesis, and the theorem will be proved.

So put $\phi(x)=F(a_1x)-F(x)$ and suppose that

$$\phi(x_1) = \beta + \epsilon > \beta. \tag{2}$$

Since $\phi(x)$ is right almost periodic let $s_1, \cdots, s_k$ be such that to every $t\in G$ we can associate an integer $i\leqq k$ for which $|\phi(xt)-\phi(xs_i)|<\epsilon/2$. In particular, for $x=x_1s_i^{-1}$: $|\phi(x_1s_i^{-1}t)-\phi(x_1)|<\epsilon/2$, so that by (2) $\phi(x_1s_i^{-1}t)>\beta+\epsilon/2$, i.e.,

$$F(a_1x_1s_i^{-1}t) - F(x_1s_i^{-1}t) > \beta + \epsilon/2. \tag{3}$$

Now consider the right almost periodic functions

$$\phi_i(x) = F(a_1x_1s_i^{-1}x) - F(x) \qquad (i = 1, \cdots, k).$$

We can extract from the sequence (c_n) a subsequence (b_n) such that $|\phi_i(xb_q^{-1}b_p)-\phi_i(x)|<\epsilon/2$, whatever be $x\in G$, b_p, b_q, and $i=1, \cdots, k$. We deduce

$$\begin{aligned} &| F(a_1x_1s_i^{-1}tb_q^{-1}b_p) - F(tb_q^{-1}b_p) - F(a_1x_1s_i^{-1}t) + F(t) | \\ &\qquad\qquad = | \phi_i(tb_q^{-1}b_p) - \phi_i(t) | < \epsilon/2. \end{aligned}$$

Hence by (1)

$$F(a_1x_1s_i^{-1}tb_q^{-1}b_p) - F(a_1x_1s_i^{-1}t) > \alpha - \epsilon/2. \tag{4}$$

(3) and (4) give, by addition, the required relation:

$$F(a_1x_1s_i^{-1}tb_q^{-1}b_p) - F(x_1s_i^{-1}t) > \alpha + \beta.$$

The proof is now complete.

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