HOMOTOPY GROUPS OF CERTAIN DELETED PRODUCT SPACES

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If X is a topological space, let D_X denote the subset of $X \times X$ consisting of the set of all points of the form (x, x), where $x \in X$. Then the deleted product space, X^*, of X is the space $X \times X - D_X$ with the relative topology. It follows from a theorem of Eilenberg (see [1, p. 43]) that for a connected, finite, 1-dimensional polyhedron X, X^* is arcwise connected if and only if X is not an arc.

In this paper we prove the following theorem:

If X is a connected, finite, 1-dimensional polyhedron which is not an arc, then $\Pi_k(X^*)=0$ for all $k>1$.

DEFINITION 1. If X is a connected, finite, 1-dimensional polyhedron and A and B are subpolyhedra of X, let $P(A \times B - D_X) = \bigcup \{ r \times s \mid r$ is a simplex of A, s is a simplex of B, and $r \cap s = \emptyset \}$.

REMARK 1. Let X be a connected, finite, 1-dimensional polyhedron which is not an arc. If X does not have a vertex of order $\geqq 3$, then X is a simple closed curve. If X does have a vertex of order $\geqq 3$, let A' be a triod in X. Then it is clear that there is a subdivision X' of X such that: (1) each simplex of A' is a simplex of X', (2) X'consists of a finite number of 1-simplexes, $r_1, \cdots, r_k$, and (3) X' can be realized by starting with A' and adding one 1-simplex r_j at a time so that either $r_j \cap (\bigcup_{k=1}^{j-1} r_k)$ is a single vertex or $r_j \cap (\bigcup_{k=1}^{j-1} r_k)$ consists of two vertices v_1, v_2, where each v_i $(i=1, 2)$ is a vertex of order one in $\bigcup_{k=1}^{j-1} r_k$. In this paper, we shall assume that such a subdivision of X has been made.

Definition 1 and Remark 1 may be found in [3]. It is shown in [4] that if X is a connected, finite, 1-dimensional polyhedron, then there is a deformation retraction of X^* onto $P(X^*)$.

DEFINITION 2. A space X is said to be *aspheric* if $\Pi_k(X)=0$ for all $k>1$.

THEOREM 1. *Let $X=A \cup B$, where X, A, and B are connected polyhedra and $A \cap B$ has a finite number of components C_i. Suppose that*

(1) *A, B, and all the C_i are aspheric.*

(2) *For each of the C_i, the injections*

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$\overset{1}{i_*}: \Pi_1(C_i) \to \Pi_1(A)$ *and*

$\overset{2}{i_*}: \Pi_1(C_i) \to \Pi_1(B)$ *are isomorphisms into.*

Then X is aspheric.

The above theorem is due to J. H. C. Whitehead [**5**, p. 159].

THEOREM 2. *If X is a connected, finite,* 1*-dimensional polyhedron which is not an arc, then X^* is aspheric.*

PROOF. The author [**3**] has shown that if X is either a simple closed curve or a triod, then X^* has the homotopy type of a simple closed curve.

The proof is by induction on the number of 1-simplexes of X. Suppose that the theorem is true if X consists of $n-1$ 1-simplexes, where $n \geqq 4$. Now suppose X is not a simple closed curve and X consists of n 1-simplexes. By Remark 1, it is possible to express X as $X = A \cup B$, where A is a connected polyhedron which is the union of $n-1$ 1-simplexes, B is a 1-simplex, and either (1) $A \cap B = \{v\}$, where v is a vertex of X, or (2) $A \cap B = \{v_1\} \cup \{v_2\}$, where v_1 and v_2 are vertices of order 2 in X. It is possible to choose A in such a manner that A contains a vertex of order $\geqq 3$, and we assume that this has been done.

CASE (1). It is easy to see that

$$P(X^*) = P(A^*) \cup P(B \times A - D_X) \cup P(A \times B - D_X)$$

and

$$P(A^*) \cap P(B \times A - D_X) = P(v \times A - D_X).$$

First we show that

$$P(A^*) \cup P(B \times A - D_X) = P(X \times A - D_X)$$

is aspheric. Let u be the other vertex of B. Then $u \times A \subset P(B \times A - D_X)$, and it is easy to see that there is a deformation retraction of $P(B \times A - D_X)$ onto $u \times A$. Therefore $P(B \times A - D_X)$ and $u \times A$ have the same homotopy type. Hence $\Pi_k(P(B \times A - D_X)) = 0$ for all $k > 1$ since $u \times A$ is a 1-dimensional polyhedron.

Let $C_1, \cdots, C_q$ denote the components of $P(v \times A - D_X)$. Each C_i is aspheric since it is a 1-dimensional polyhedron. For each C_i, consider the diagram

$$\Pi_1(C_i) \xrightarrow{\overset{1i}{j_*}} \Pi_1(v \times A) \xrightarrow{\overset{2}{j_*}} \Pi_1(A \times A)$$

where j_*^{1i} and j_*^2 are the injection homomorphisms. It is clear that j_*^{1i} is an isomorphism into since $C_i = v \times A_i$, where A_i is a subpolyhedron of A, and it is a well-known result that j_*^2 is an isomorphism into. Therefore $j_*^2 j_*^{1i}$ is an isomorphism into.

Now consider the diagram

$$\Pi_1(C_i) \xrightarrow{k_*^{1i}} \Pi_1(P(A^*)) \xrightarrow{j_*^3} \Pi_1(A \times A)$$

where k_*^{1i} and j_*^3 are injection homomorphisms. By the above argument $k_*^{1i} j_*^3$ is an isomorphism into. Hence k_*^{1i} is an isomorphism into.

Now consider the diagrams

$$\Pi_1(C_i) \to \Pi_1(v \times A) \to \Pi_1(B \times A),$$

$$\Pi_1(C_i) \xrightarrow{k_*^{2i}} \Pi_1(P(B \times A - D_X)) \to \Pi_1(B \times A)$$

where each indicated homomorphism is the injection homomorphism. By essentially repeating the above argument with respect to the new diagrams, it can be shown that k_*^{2i} is an isomorphism into.

Therefore the conditions of Theorem 1 are satisfied, and hence $P(X \times A - D_X)$ is aspheric.

Now we consider

$$P(X \times A - D_X) \cup P(A \times B - D_X) = P(X^*).$$

Observe that

$$P(X \times A - D_X) \cap P(A \times B - D_X) = P(A \times v - D_X).$$

Let $D_1, \cdots, D_q$ denote the components of $P(A \times v - D_X)$. Again $P(A \times B - D_X)$ and D_i $(i = 1, \cdots, q)$ are aspheric. In order to show that $P(X^*)$ is aspheric, we consider the four diagrams

$$\Pi_1(D_i) \to \Pi_1(A \times v) \to \Pi_1(X \times A),$$

$$\Pi_1(D_i) \xrightarrow{m_*^{1i}} \Pi_1(P(X \times A - D_X)) \longrightarrow \Pi_1(X \times A),$$

$$\Pi_1(D_i) \to \Pi_1(A \times v) \to \Pi_1(A \times B),$$

$$\Pi_1(D_i) \xrightarrow{m_*^{2i}} \Pi_1(P(A \times B - D_X)) \longrightarrow \Pi_1(A \times B)$$

where each indicated homomorphism is the injection homomorphism.

Using these four diagrams, we essentially repeat the above argument to show that m_*^{1i} and m_*^{2i} are isomorphisms into. Then, by Theorem 1, $P(X^*)$ is aspheric.

CASE (2). It is easy to see that

$$P(X^*) = P(A^*) \cup P(B \times A - D_X) \cup P(A \times B - D_X)$$

and

$$P(A^*) \cap P(B \times A - D_X) = P(v_1 \times A - D_X) \cup P(v_2 \times A - D_X).$$

First we show that

$$P(A^*) \cup P(B \times A - D_X) = P(X \times A - D_X)$$

is aspheric. For $i=1, 2$, $P(v_i \times A - D_X)$ is a connected, 1-dimensional polyhedron, and hence it is aspheric. Let A' be the subpolyhedron of A consisting of the union of all the 1-simplexes in A except those two which have v_1 and v_2 as vertices. Then $v_1 \times A' \subset P(B \times A - D_X)$, and it is easy to see that there is a deformation retraction of $P(B \times A - D_X)$ onto $v_1 \times A'$. Hence $\Pi_k(P(B \times A - D_X)) = 0$ for all $k > 1$ since $v_1 \times A'$ is a connected, 1-dimensional polyhedron.

Consider the diagram

$$\Pi_1(P(v_i \times A - D_X)) \xrightarrow{k_*^{1i}} \Pi_1(P(A^*)) \xrightarrow{j_*^1} \Pi_1(A \times A)$$

where k_*^{1i} and j_*^1 are the injection homomorphisms. Now $j_*^1 k_*^{1i}$ is an isomorphism into since $P(v_i \times A - D_X)$ and $v_i \times A$ have the same homotopy type. Therefore k_*^{1i} is an isomorphism into. By considering a similar diagram, it is easy to see that the injection homomorphism

$$k_*^{2i} : \Pi_1(P(v_i \times A - D_X)) \to \Pi_1(P(B \times A - D_X))$$

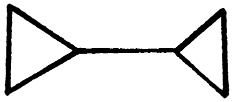
is an isomorphism into. Hence the conditions of Theorem 1 are satisfied, and therefore $P(X \times A - D_X)$ is aspheric.

Now we consider

$$P(X \times A - D_X) \cup P(A \times B - D_X) = P(X^*).$$

By essentially repeating the above argument, it can be shown that $P(X^*)$ is aspheric.

EXAMPLE. Let X be the polyhedron shown in the following diagram:



Then $H_2(X^*, Z)$ is a free abelian group of rank 2 [2, p. 364]. Therefore X^* does not admit a 1-complex as deformation retract. This answers in the negative a question asked by the referee.

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