Case (5) and (6) continuous spectrum $0 < \lambda < \infty$, point spectrum $-\infty < \lambda < 0$,

Case (7) and (8) point spectrum $0 < \lambda < \infty$, continuous spectrum $-\infty < \lambda < 0$,

Case (9) point spectrum $Q_1 < \lambda < Q_2$, continuous spectrum $\lambda < \min[Q_1, Q_2]$, $\lambda > \max[Q_1, Q_2]$.

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A METHOD OF APPROXIMATING THE ZEROS OF FUNCTIONS BY QUADRATIC FORMULAS[1]

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1. Introduction. The problem of approximating two zeros of a given function by solving a quadratic equation was discussed in a number of papers [**1**; **2**; **4**; **5**; **7**]. In this paper we present a general method of deriving quadratic equations the two roots of which would approximate two zeros of an analytic function $f(z)$. A function $f(z)/g(z, u)$, instead of $f(z)$, is considered, where $g(z, u)$ is another appropriately chosen analytic function. By varying the parameter u, and keeping the initial approximation to the zeros unchanged, the final approximations can be improved or another pair of zeros approximated. The exact values of the zeros are determined as limits of the expressions approximating them.

2. The general method. Let $f(z)$ and $g(z, u)$ be analytic functions

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within and on the circle C, where $f(z)$ has zeros, simple or multiple. The function $g(z, u)$ may have zeros in common with $f(z)$. However, we assume that those zeros of $f(z)$ which we desire to calculate are simple zeros of $f(z)/g(z, u)$.

Let the expansion of $g(z, u)/f(z)$ into partial fractions be

$$g/f = \sum A_j^{(1)}/h_j^{m_j} + \psi_1, \tag{1}$$

where $g=g(z, u)$, $f=f(z)$, $h_j=z-a_j$, A_j is a polynomial in $z-a_j$ of a degree not higher than m_j-1, m_j being the multiplicity of the zero a_j; $\psi_1=\psi_1(z)$ is analytic within and on C; and the summation is carried out over all a_j.

On differentiating (1) $(n-1)$ times and multiplying it by $(-1)^{n-1}/(n-1)!$, we obtain

$$H_n = \sum A_j^{(n)}/h_j^{n+m_j-1} + \psi_n, \tag{2}$$

where $A_j^{(n)}$ again is a polynomial in $z-a_j$ of a degree not higher than m_j-1, $\psi_n=(-1)^{n-1}\psi_1^{(n-1)}/(n-1)!$, and $H_n=Q_n/f^n$. The function Q_n is the determinant

$$Q_n = \begin{vmatrix} g & f & 0 & \cdots \\ g' & f' & f & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ g^{(n-1)}/(n-1)! & f^{(n-1)}/(n-1)! & f^{(n-2)}/(n-2)! & \cdots f' \end{vmatrix}. \tag{3}$$

It can be evaluated recursively, [3],

$$\begin{aligned} Q_n &= f'Q_{n-1} - ff''Q_{n-2}/2! + \cdots + (-f)^{n-2}f^{(n-1)}Q_1/(n-1)! \\ &\quad + (-f)^{n-1}g^{(n-1)}Q_0/(n-1)! \qquad n = 3, 4, \cdots, \\ Q_0 &= 1, \qquad Q_1 = g, \qquad Q_2 = f'Q_1 - ff''Q_0. \end{aligned} \tag{4}$$

We assume that a_1 and a_2 are the two simple zeros of $f(z)/g(z, u)$ which we wish to calculate and rewrite (2) in the form

$$H_n = (A_1/h_1^n + A_2/h_2^n)(1 + \beta_n), \tag{5}$$

where

$$\beta_n = h_1^n h_2^n\left(\sum{}' A_j^{(n)}/h_j^{n+m_j-1} + \psi_n\right)/(A_1h_2^n + A_2h_1^n); \tag{6}$$

the prime on the summation sign means that the terms for $j=1, 2$, are to be omitted, and A_1, A_2 stand for $A_1^{n+m_1-1}$ and $A_2^{n+m_2-1}$.

Now if u is a fixed number and z is given a numerical value such that

$$
(7) \qquad \begin{aligned} |h_1| &\leqq |h_2| < |h_j|, \qquad j = 3, 4, \cdots, \\ |h_1| &\leqq |h_2| < |v - z|, \end{aligned}
$$

for any v on C, then

$$
(8) \qquad \lim \beta_n = 0 \quad \text{as} \quad n \to \infty.
$$

Using (5) we may write the following equations:

$$
(9) \qquad \begin{aligned} A_1/h_1^{n-1} + A_2/h_2^{n-1} &= (1 + \alpha_{n-1})H_{n-1}, \\ A_1/h_1^{n} + A_2/h_2^{n} &= (1 + \alpha_n)H_n, \\ A_1/h_1^{n+1} + A_2/h_2^{n+1} &= (1 + \alpha_{n+1})H_{n+1}, \end{aligned}
$$

and, eliminating A_1/h_1^{n-1} and A_2/h_2^{n-2} between them, obtain

$$
(10) \quad (1 + \alpha_{n-1})H_{n-1} - (1 + \alpha_n)(h_1 + h_2)H_n + (1 + \alpha_{n+1})h_1h_2H_{n+1} = 0,
$$

where

$$
(11) \qquad \lim \alpha_n = 0 \quad \text{as} \quad n \to \infty.
$$

Equation (10) shows that h_1 and h_2 satisfy a quadratic equation. Therefore, we may write

$$
(12) \qquad \begin{aligned} h^2 + ph + q &= 0, \\ \overset{1}{H}_{n-1} + p\overset{1}{H}_n + q\overset{1}{H}_{n+1} &= 0, \\ \overset{1}{H}_n + p\overset{1}{H}_{n+1} + q\overset{1}{H}_{n+2} &= 0, \end{aligned}
$$

where

$$
(13) \qquad p = -(h_1 + h_2), \quad q = h_1h_2, \quad \text{and} \quad \overset{1}{H}_j = (1 + \alpha_j)H_j, \qquad j = n-1, n, n+1, n+2.
$$

Equating the determinant of (12) to zero, we get the following equation which is satisfied by h_1 and h_2:

$$
(14) \qquad \begin{vmatrix} h^2 & h & 1 \\ \overset{1}{H}_{n-1} & \overset{1}{H}_n & \overset{1}{H}_{n+1} \\ \overset{1}{H}_n & \overset{1}{H}_{n+1} & \overset{1}{H}_{n+2} \end{vmatrix} = 0.
$$

The equation approximating h_1 and h_2 may now be written,

$$
(15) \qquad \begin{vmatrix} h^2 & h & 1 \\ H_{n-1} & H_n & H_{n+1} \\ H_n & H_{n+1} & H_{n+2} \end{vmatrix} = 0,
$$

or

$$\begin{vmatrix} h^2 & fh & f^2 \\ Q_{n-1} & Q_n & Q_{n+1} \\ Q_n & Q_{n+1} & Q_{n+2} \end{vmatrix} = 0. \tag{16}$$

Note that we have not introduced a separate symbol for the approximating value of h.

From (14) also follows

$$a_1, a_2 = z - f \lim_{n\to\infty} B_n \pm (B_n - 4A_nA_{n-1})^{1/2}/2A_n, \tag{17}$$

where

$$A_n = Q_nQ_{n+2} - Q_{n+1}^2, \qquad B_n = Q_{n-1}Q_{n+2} - Q_nQ_{n+1}. \tag{18}$$

The quadratic equation (16) is satisfied exactly by the zeros of a quadratic polynomial in z if $g(z, u)$ is a polynomial in z. If the degree of $g(z, u)$ is k, it is satisfied starting with the lowest $Q_{n-1} = Q_k$. This follows from the fact that (16) would coincide with (14) starting with $n-1=k$.

We consider in more detail (16) when $g(z, u)=1$. In this case the determinant (3) is reduced to a determinant of order $n-1$. We denote it by P_{n-1} and write the recursive relation between P_n and the lower determinants.

$$P_n = f'P_{n-1} - ff''P_{n-2}/2! + \cdots + (-1)^{n-1}f^{(n)}P_0/n!, \quad P_0 = 1. \tag{19}$$

The equation corresponding to (16), with n increased by one, we rewrite in the form

$$\begin{vmatrix} P_{n+1} & P_n \\ P_{n+2} & P_{n+1} \end{vmatrix} h^2 - \begin{vmatrix} P_n & P_{n-1} \\ P_{n+2} & P_{n+1} \end{vmatrix} fh + \begin{vmatrix} P_n & P_{n-1} \\ P_{n+1} & P_n \end{vmatrix} f^2 = 0. \tag{20}$$

There is no difficulty in showing that

$$\begin{vmatrix} P_n & P_{n-1} \\ P_{n+1} & P_n \end{vmatrix} = f^nK_n \tag{21}$$

and

$$\begin{vmatrix} P_n & P_{n-1} \\ P_{n+2} & P_{n+1} \end{vmatrix} = f^nL_{n+1}, \qquad L_{n+1} = f'K_n - fM_n, \tag{22}$$

where

$$(23)\quad K_n = \begin{vmatrix} f''/2! & f' & f & \cdots & . \\ f'''/3! & f''/2! & f' & \cdots & . \\ . & . & . & \cdots & . \\ f^{(n+1)}/(n+1)! & f^{(n)}/n! & f^{(n-1)}/(n-1)! & \cdots & f''/2! \end{vmatrix},$$

$$(24)\quad M_n = \begin{vmatrix} f'''/3! & f' & f & \cdots & . \\ f^{\text{iv}}/4! & f''/2! & f' & \cdots & . \\ . & . & . & \cdots & . \\ f^{(n+2)}/(n+2)! & f^{(n)}/n! & f^{(n-1)}/(n-1)! & \cdots & f''/2! \end{vmatrix},$$

$$(25)\quad L_n = \begin{vmatrix} f' & f & 0 & 0 & \cdots & 0 \\ f'''/3! & f''/2! & f' & f & \cdots & 0 \\ f^{\text{iv}}/4! & f'''/3! & f''/2! & f' & \cdots & . \\ . & . & . & . & \cdots & . \\ f^{(n+2)}/(n+2)! & f^{(n+1)}/(n+1)! & f^{(n)}/n! & f^{(n-1)}/(n-1)! & \cdots & f''/2! \end{vmatrix}.$$

With these notations (20) can now be presented in a shorter form,

$$(26)\qquad K_{n+1}h^2 - L_{n+1}h + fK_n = 0, \text{ or}$$

$$(27)\qquad K_{n+1}h^2 + (fM_n - f'K_n)h + fK_n = 0.$$

Further simplifications of (26) or (27) are possible in some particular cases. As an illustration, we take the cubic trinomial z^3+pz+q.

For $z=0$, $f=q$, $f'=p$, $f''/2!=0$, $f'''/3!=1$, $f^{(n)}=0$, $n=4, 5, \cdots$, and it may be seen at once that

$$(28)\qquad K_n = -pK_{n-2} + qK_{n-3}$$

and

$$(29)\qquad L_{n+1} = -K_{n+2}.$$

Thus, the two zeros of z^3+pz+q which are smaller in absolute value can be approximated by solving

$$(30)\qquad K_{n+1}h^2 + K_{n+2}h + qK_n = 0.$$

The function K_n can be evaluated recursively by using (28) or by solving (28) for K_n in terms of p and q (see [6]). The solution for K_n is:

$$(31)\qquad \begin{aligned} &K_n = (-p)^r\left[1 + \sum_{k=1}^{m}(-1)^k\binom{r-k}{2k}X^k\right] \qquad \text{for } n \text{ even;} \\ &r = n/2, \qquad m = [n/6], \qquad X = q^2/p^3; \end{aligned}$$

$$(32)\qquad \begin{aligned} &K_n = (-p)^{r-1}q\sum_{k=0}^{m}(-1)^k\binom{r-k}{2k+1}X^k \qquad \text{for } n \text{ odd,} \\ &r = (n-1)/2, \qquad m = [(n-2)/6], \qquad X = q^2/p^3. \end{aligned}$$

The first few values of K_n are

$$K_0 = 1,\ K_1 = 0,\ K_2 = -p,\ K_3 = q,\ K_4 = p^2,\ K_5 = -2pq,$$
$$K_6 = -p^3 + q^2,\ K_7 = 3p^2q,\ K_8 = p^4 - 3pq^2,\ K_9 = -4p^3q + q^3,$$
$$K_{10} = -p^5 + 6p^2q^2,\ K_{11} = 5p^4q - 4pq^3,\ K_{12} = p^6 - 10p^3q^2 + q^4.$$

We will be using another simple case in which (16) is satisfied exactly by the zeros of a quadratic polynomial, namely, $g(z, u) = f'(z)$. Using the notation D_n for Q_n the recursive formula is

$$\begin{aligned} D_n &= f'D_{n-1} - ff''D_{n-2}/2! + \cdots + (-f)^{n-2}f^{(n-1)}D_1/(n-1)! \\ &\quad + (-f)^{n-1}f^{(n)}D_0/(n-1)!, \qquad n = 3, 4, \cdots, \end{aligned} \tag{33}$$

$$D_0 = 1, \qquad D_1 = f', \qquad D_2 = f'^2 - ff''.$$

3. **The use of the parameter.** The usefulness of introducing a parameter into quadratic formulas will now be illustrated.

Let

$$q(z, u) = (u - z)^k f'(z), \tag{34}$$

where k is a positive integer and u an arbitrary number not equal to z, a_1, or a_2. The zeros a_1 and a_2 may be simple or multiple. By applying (5) we get

$$H_{n,k} = (m_1 e_1^k/h_1^n + m_2 e_2^k/h_2^n)(1 + \beta_{n,k}), \tag{35}$$

where $h_1 = z - a_1$, $h_2 = z - a_2$, as before; $e_1 = u - a_1$, $e_2 = u - a_2$; m_1 and m_2 are the multiplicities of a_1 and a_2 respectively; $H_{n,k} = H_{n,k}(z, u)$, $\beta_{n,k} = \beta_{n,k}(z, u)$.

Now if we assume that $n - k =$ constant and z and u are given such numerical values that

$$|e_1/h_1| \geqq |e_2/h_2| > |e_j/h_j|, \qquad j = 3, 4, \cdots, \tag{36}$$

and

$$|e_1/h_1| \geqq |e_2/h_2| > |(u - v)/(z - v)|,$$

for any v on C, then $\lim \beta_{n,k} = 0$ as $n \to \infty$.

Let $n - k = 0$; then we can write the approximate equations

$$\begin{aligned} m_1(e_1/h_1)^{n-1} + m_2(e_2/h_2)^{n-1} &= H_{n-1,n-1}, \\ m_1(e_1/h_1)^{n} + m_2(e_2/h_2)^{n} &= H_{n,n}, \\ m_1(e_1/h_1)^{n+1} + m_2(e_2/h_2)^{n+1} &= H_{n+1,n+1} \end{aligned} \tag{37}$$

and, eliminating $m_1(e_1/h_1)^{n-1}$ and $m_2(e_2/h_2)^{n-1}$, we obtain

$$H_{n-1,n-1} + pH_{n,n} + qH_{n+1,n+1} = 0, \tag{38}$$

where $p = -(h_1/e_1 + h_2/e_2)$, $q = h_1h_2/e_1e_2$.

We write now the equations

$$\begin{aligned} (h/e)^2 + ph/e + q &= 0, \\ H_{n-1,n-1} + pH_{n,n} + qH_{n+1,n+1} &= 0, \\ H_{n,n} + pH_{n+1,n+1} + qH_{n+2,n+2} &= 0 \end{aligned} \tag{39}$$

and, eliminating p and q between them, obtain the desired quadratic equation

$$\begin{vmatrix} (h/e)^2 & h/e & 1 \\ H_{n-1,n-1} & H_{n,n} & H_{n+1,n+1} \\ H_{n,n} & H_{n+1,n+1} & H_{n+2,n+2} \end{vmatrix} = 0 \tag{40}$$

or

$$\begin{vmatrix} (z-a)^2/(u-a)^2 & (z-a)/(u-a)f & f^2 \\ Q_{n-1,n-1} & Q_{n,n} & Q_{n+1,n+1} \\ Q_{n,n} & Q_{n+1,n+1} & Q_{n+2,n+2} \end{vmatrix} = 0. \tag{41}$$

Another pair of simple quadratic equations will be obtained by starting with

$$\begin{aligned} m_1e_1^k/h_1^{n-1} + m_2e_2^k/h_2^{n-1} &= H_{n-1,k}, \\ m_1e_1^k/h_1^n + m_2e_2^k/h_2^n &= H_{n,k}, \\ m_1e_1^k/h_1^{n+1} + m_2e_2^k/h_2^{n+1} &= H_{n+1,k} \end{aligned} \tag{42}$$

and

$$\begin{aligned} m_1e_1^{k-1}/h_1^n + m_2e_2^{k-1}/h_2^n &= H_{n,k-1}, \\ m_1e_1^k/h_1^n + m_2e_2^k/h_2^n &= H_{n,k}, \\ m_1e_1^{k+1}/h_1^n + m_2e_2^{k+1}/h_2^n &= H_{n,k+1}, \end{aligned} \tag{43}$$

namely,

$$\begin{vmatrix} (z-a)^2 & (z-a)f & f^2 \\ Q_{n-1,k} & Q_{n,k} & Q_{n+1,k} \\ Q_{n,k} & Q_{n+1,k} & Q_{n+2,k} \end{vmatrix} = 0 \tag{44}$$

and

$$\begin{vmatrix} 1 & (u-a)f & (u-a)^2f^2 \\ Q_{n,k-1} & Q_{n,k} & Q_{n,k+1} \\ Q_{n,k} & Q_{n,k+1} & Q_{n,k+2} \end{vmatrix} = 0. \tag{45}$$

The function $Q_{n,k} = Q_{n,k}(z, u)$ may be evaluated by (3) or (4). However, it is more practical to use its expression in terms of D_j (see [3]),

$$Q_{n,k} = \sum_{j=0}^{n-1} \binom{k}{j} (u-z)^{k-j} P_{n-1-j}, \tag{46}$$

which can be derived from (3) or (4) or in some other way.

If $g(z, u) = (u-z)^k$, then

$$Q_{n,k} = \sum_{j=0}^{n-1} \binom{k}{j} (u-z)^{k-j} P_{n-1-j} \tag{47}$$

and the equations similar to (40), (43) and (44) can be derived in the same way.

The selection of the parameter u is not very difficult in many practical cases; however, it is a complicated problem in the general case.

For an illustration we again take the cubic trinomial z^3+pz+q. If its two zeros are imaginary, they are larger in absolute value than the real one when $p>0$. Therefore, they cannot be calculated by (30), as in the previous illustration, but they can be calculated by (40), (43), or (44) with $z=0$, as before, and $u=-q/p$. It is a simple matter to show that the inequalities (36) are satisfied and the coefficients of the corresponding quadratic equations are expressible in terms of p and q.

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