THE CIRCUMFERENCE OF A CONVEX POLYGON

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In this note we combine a convexity theorem due to Cauchy with a combinatorial identity discovered by M. Kac.

Cauchy's theorem [1] concerns the length L of the circumference of a compact convex set A in the plane. Let $D(\theta)$ denote the projection of A on a line with direction θ, $0 \leq \theta < \pi$, or, if $z = x + iy$

$$D(\theta) = \max_{z \in A} (x \cos \theta + y \sin \theta) - \min_{z \in A} (x \cos \theta + y \sin \theta). \tag{1}$$

Then

$$L = \int_0^\pi D(\theta) d\theta. \tag{2}$$

M. Kac, in [2], considered a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ with real components. For each permutation

$$\sigma = \begin{Bmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{Bmatrix}$$

he defined the vectors

$$\mathbf{x}(\sigma) = (x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}),$$

and their partial sums

$$s_0(\sigma) = 0, \qquad s_k(\sigma) = x_{\sigma_1} + x_{\sigma_2} + \cdots + x_{\sigma_k}, \qquad k = 1, 2, \cdots, n.$$

His result may be stated in the form

$$\sum_\sigma \left[\max_{0 \leq k \leq n} s_k(\sigma) - \min_{0 \leq k \leq n} s_k(\sigma) \right] = \sum_\sigma \sum_{k=1}^n \frac{1}{k} \mid s_k(\sigma) \mid \tag{3}$$

where the σ-summation extends over the group of all permutations of n-objects.

We shall consider a vector $z = (z_1, z_2, \cdots, z_n)$ with complex components. As above we let

$$z(\sigma) = (z_{\sigma_1}, z_{\sigma_2}, \cdots, z_{\sigma_n}),$$
$$s_0(\sigma) = 0, \qquad s_k(\sigma) = z_{\sigma_1} + z_{\sigma_2} + \cdots + z_{\sigma_k}, \qquad k = 1, 2, \cdots, n.$$

Received by the editors May 14, 1960 and, in revised form, July 8, 1960.

[1] Research supported by the Office of Naval Research at the University of Minnesota.

We define the set $A(\sigma)$ as the smallest convex set (polygon) containing all the points $s_0(\sigma), s_1(\sigma), \cdots, s_n(\sigma)$, and $L(\sigma)$ as the length of the circumference of $A(\sigma)$. We shall obtain the following generalization of equation (3).

THEOREM 1.

$$\sum_\sigma L(\sigma) = 2\sum_\sigma \sum_{k=1}^n \frac{1}{k} \,|\, s_k(\sigma) \,|\, . \tag{4}$$

Here the summation again extends over all permutations, and (4) is equivalent to (3) when all z_k are real. When the z_k are not real we write

$$\begin{aligned} z_k &= x_k + iy_k, \quad z_{\sigma_k} = x_{\sigma_k} + iy_{\sigma_k}, \\ t_k(\theta) &= x_k \cos\theta + y_k \sin\theta, \qquad\qquad 0 \leqq \theta < \pi, \\ u_0(\theta, \sigma) &= 0, \quad u_k(\theta,\sigma) = t_{\sigma_1}(\theta) + \cdots + t_{\sigma_k}(\theta), \quad k = 1, 2, \cdots, n. \end{aligned}$$

Let $D(\theta, \sigma)$ be the projection of $A(\sigma)$ on a line with direction θ. Since $A(\sigma)$ is the convex hull of its extreme points we have from (1)

$$D(\theta,\sigma) = \max_{0\leqq k\leqq n} u_k(\theta,\sigma) - \min_{0\leqq k\leqq n} u_k(\theta,\sigma).$$

By equation (2)

$$L(\sigma) = \int_0^\pi \left[\max_{0\leqq k\leqq n} u_k(\theta,\sigma) - \min_{0\leqq k\leqq n} u_k(\theta,\sigma) \right] d\theta.$$

By equation (3)

$$\sum_\sigma L(\sigma) = \sum_\sigma \sum_{k=1}^n \frac{1}{k} \int_0^\pi |\, u_k(\theta,\sigma) \,|\, d\theta.$$

But

$$\int_0^\pi |\, u_k(\theta,\sigma)\,|\, d\theta = 2\left[\left\{ \sum_{i=1}^k x_{\sigma_i} \right\}^2 + \left\{ \sum_{i=1}^k y_{\sigma_i} \right\}^2 \right]^{1/2} = |\, s_k(\sigma) \,|\, .$$

Hence (4) is proved.

As an application we derive a result of probabilistic interest. Let $Z_1, Z_2, \cdots$ denote a sequence of identically distributed independent complex valued random variables. Thus the distribution of each Z_k is the same planar Lebesgue-Stieltjes measure and their joint distributions are given by the obvious product measure. We define their partial sums as the random variables

$$S_0 = 0, \qquad S_n = Z_1 + Z_2 + \cdots + Z_n, \qquad n \geqq 1.$$

Finally, let L_n be defined as the length of the circumference of the smallest convex set containing $S_0, S_1, \cdots, S_n$. If "E" denotes expectation with respect to the product measure, we have

THEOREM 2.

$$E(L_n) = 2 \sum_{k=1}^{n} \frac{1}{k} E|S_k|. \tag{5}$$

The proof is based on two observations. First, L_n is a continuous function of $S_1, S_2, \cdots, S_n$, so that it is a random variable. Secondly, if we define the random variables Z_{σ_k}, $S_k(\sigma)$, $L_n(\sigma)$ as was done in the deterministic case, it follows from the invariance of the product measure under permutations σ that the expectations $E|S_n(\sigma)|$ and $E(L_n(\sigma))$ are independent of σ. This proves the theorem and also shows that either both sides in (5) are finite or neither. Of course they are finite if and only if $E|Z_1| < \infty$.

Finally, we consider two situations where the asymptotic behavior of $E(L_n)$ is of some interest.

(a) Let $Z_1, Z_2, \cdots$ be identically distributed and independent with

$$Z_k = X_k + iY_k, \qquad E(X_k) = E(Y_k) = 0,$$
$$E(X_k^2) = a^2 < \infty, \qquad E(Y_k^2) = b^2 < \infty, \qquad E(X_kY_k) = \rho ab.$$

Then $n^{-1/2}(Z_1 + \cdots + Z_n)$ has a bivariate normal limiting distribution and its absolute value may be shown to be uniformly integrable in n, so that

$$\lim_{n\to\infty} n^{-1/2}E|Z_1 + \cdots + Z_n|$$
$$= (2\pi)^{-1/2} \int_0^{\pi} [a^2 \sin^2\theta + b^2\cos^2\theta + 2ab\rho\sin\theta\cos\theta]^{1/2} d\theta = c.$$

It follows from Theorem 2 that

$$\lim_{n\to\infty} n^{-1/2}E(L_n) = 4c. \tag{6}$$

(b) Here we let $X_1, X_2, \cdots$ be a sequence of identically distributed independent random variables with

$$E(X_k) = \mu, \qquad E[(X_k - \mu)^2] = \sigma^2 < \infty.$$

We define the complex valued random variables

$$Z_k = X_k + i$$

and their partial sums

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n + ni, \qquad n = 1, 2, \cdots.$$

The law of large numbers asserts that $n^{-1}S_n \to \mu + i$ with probability one. Geometrically this means that the polygonal path consisting of the points $S_0, S_1, \cdots$ does not deviate too far from the straight line through 0 and $\mu + i$. We shall denote by L_n the circumference of the smallest convex set containing the points $S_0, S_1, \cdots, S_n$, and quite naturally, study the excess of L_n over its smallest possible value, which is $2n(1+\mu^2)^{1/2}$.

Theorem 2 yields

$$\frac{1}{2} E(L_n) - n(1 + \mu^2)^{1/2} = \sum_{k=1}^{n} E\left[\left\{\left(\frac{X_1 + X_2 + \cdots + X_k}{k}\right)^2 + 1\right\}^{1/2} - (\mu^2 + 1)^{1/2}\right].$$

Using the second order Taylor expansion of $(t^2+1)^{1/2}$ about $t=\mu$, it is quite simple to show that, as $k \to \infty$,

$$E\left[\left\{\left(\frac{X_1 + \cdots + X_k}{k}\right)^2 + 1\right\}^{1/2} - (\mu^2 + 1)^{1/2}\right] \sim \frac{1}{2}(1 + \mu^2)^{-3/2} E\left[\left(\frac{X_1 + \cdots + X_k - \mu}{k}\right)^2\right] = \frac{1}{2}(1 + \mu^2)^{-3/2}\frac{\sigma^2}{k}.$$

It follows that

$$\lim_{n\to\infty} (\log n)^{-1}[E(L_n) - 2n(1 + \mu^2)^{1/2}] = \sigma^2(1 + \mu^2)^{-3/2}. \tag{7}$$

References

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2. M. Kac, *Toeplitz matrices, translation kernels, and a related problem in probability theory*, Duke Math. J. vol. 21 (1954) pp. 501–509.

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