

A NOTE ON A THEOREM OF FUCHS

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L. Fuchs has shown that if G is a subdirect product of the Abelian groups G_1 and G_2 with kernels $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$, then $G_1 \times G_2 / G$ is isomorphic to $G/H_1 \times H_2$ [1].² It is natural to ask if this isomorphism holds for groups other than Abelian ones. The purpose of this note is to show that it holds whenever G is normal in $G_1 \times G_2$. We shall need the following lemma.³

LEMMA 1. *Let G be a subdirect product of the groups G_1 and G_2 . Then G is normal in $G_1 \times G_2$ if and only if G contains the derived group $(G_1 \times G_2)'$.*

PROOF. Suppose that G is normal in $G_1 \times G_2$, and let (x_1, e) and (y_1, e) be any two elements of G_1 . Since G is a subdirect product, there is an (e, g_2) in G_2 such that (y_1, g_2) is in G . And since G is normal, $(x_1, e)(y_1, g_2)(x_1, e)^{-1} = (x_1 y_1 x_1^{-1}, g_2)$ is in G . Therefore $(x_1 y_1 x_1^{-1}, g_2) \cdot (y_1, g_2)^{-1} = (x_1 y_1 x_1^{-1} y_1^{-1}, e)$ is in G , and we have $G \supseteq G_1'$. Similarly $G \supseteq G_2'$, so that $G \supseteq G_1' \times G_2' = (G_1 \times G_2)'$.

The converse is trivial since it is true in general that any subgroup which contains the derived group is normal.

THEOREM 2. *Let G be a subdirect product of G_1 and G_2 with kernels $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$. If G is normal in $G_1 \times G_2$, then $G_1 \times G_2 / G$ is isomorphic to $G/H_1 \times H_2$.*

PROOF. Since $G/H_1 \times H_2$ is isomorphic to G_1/H_1 [1, Theorem 4], we need only prove that $G_1 \times G_2 / G$ is isomorphic to G_1/H_1 .

Let $(g_1, g_2)G$ be any coset of G in $G_1 \times G_2$. There is an element (x_1, e) in G_1 such that (x_1, g_2) is in G . Define the mapping π on $G_1 \times G_2 / G$ by

$$\pi: (g_1, g_2)G \rightarrow (g_1 x_1^{-1}, e)H_1.$$

It is readily shown that π is independent of the choice of (x_1, e) in G_1 and of the choice of the representative (g_1, g_2) , and that π is a one-to-one mapping from $G_1 \times G_2 / G$ to G_1/H_1 .

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² $G_1 \times G_2$ denotes the direct product of G_1 and G_2 .

³ This lemma is equivalent to a result of Remak [2, Corollary 1 to Theorem 2].

We show now that π preserves multiplication. Given two cosets $(g_1, g_2)G$ and $(h_1, h_2)G$, we choose (x_1, e) and (y_1, e) in G_1 such that (x_1, g_2) and (y_1, h_2) belong to G . Then (x_1y_1, g_2h_2) is in G . By the lemma, $G \cap G_1 = H_1$ contains the derived group of G_1 so that G_1/H_1 is Abelian. Using this fact, we have that

$$\begin{aligned} \pi[(g_1, g_2)G(h_1, h_2)G] &= \pi[(g_1h_1, g_2h_2)G] \\ &= (g_1h_1(x_1y_1)^{-1}, e)H_1 \\ &= (g_1x_1^{-1}, e)H_1(h_1y_1^{-1}, e)H_1 \\ &= \pi[(g_1, g_2)G]\pi[(h_1, h_2)G]. \end{aligned}$$

This completes the proof of the theorem.

It is evident from these two theorems that normality imposes much stronger conditions on a subdirect product than it does on subgroups in general. One might even suspect that normal subdirect products are always trivial, i.e., are always equal to the direct product. We shall show that this is not the case. The example below gives a pair of groups which has subdirect products which are nontrivial and normal in the direct product and also has subdirect products which are not normal in the direct product.

EXAMPLE 3. Let S_i denote the symmetric group on $[a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}]$ for $i = 1, 2, 3, 4$, and put $G_1 = S_1 \times S_2$, $G_2 = S_3 \times S_4$. The alternating group A_i and the fours-group

$$V_i = [e, (a_{i,1}a_{i,2})(a_{i,3}a_{i,4}), (a_{i,1}a_{i,3})(a_{i,2}a_{i,4}), (a_{i,1}a_{i,4})(a_{i,2}a_{i,3})]$$

are normal subgroups of S_i . The derived group of S_i is A_i . In the following subdirect products of G_1 and G_2 , H_i denotes the kernel which is contained in G_i .

(i) The group $K_1 = [H_1 \times H_2, (a_{1,1}a_{1,2})(a_{3,1}a_{3,2})(H_1 \times H_2)]$ where $H_1 = A_1 \times S_2$, $H_2 = A_3 \times S_4$ is a subdirect product of index 2 in $G_1 \times G_2$ and is therefore normal.

(ii) Let $H_1 = V_1 \times S_2$, $H_2 = V_3 \times S_4$. Then the subgroup K_2 with the elements listed below is a subdirect product of index 6 in $G_1 \times G_2$:

$$\begin{aligned} H_1 \times H_2 & \quad (a_{1,2}a_{1,3})(a_{3,2}a_{3,3})(H_1 \times H_2) \\ (a_{1,1}a_{1,2})(a_{3,1}a_{3,2})(H_1 \times H_2) & \quad (a_{1,1}a_{1,2}a_{1,3})(a_{3,1}a_{3,2}a_{3,3})(H_1 \times H_2) \\ (a_{1,1}a_{1,3})(a_{3,1}a_{3,3})(H_1 \times H_2) & \quad (a_{1,1}a_{1,3}a_{1,2})(a_{3,1}a_{3,3}a_{3,2})(H_1 \times H_2). \end{aligned}$$

It is easily verified that K_2 is not normal in $G_1 \times G_2$. For example,

$$(a_{1,1}a_{1,3})^{-1}(a_{1,1}a_{1,2})(a_{3,1}a_{3,2})(a_{1,1}a_{1,3}) = (a_{1,2}a_{1,3})(a_{3,1}a_{3,2})$$

is not in K_2 .

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**FREE-ALGEBRAIC CHARACTERIZATIONS OF PRIMAL
AND INDEPENDENT ALGEBRAS**

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In the study of many-valued logics, one is led to consider a (finitary) algebra (A, o_1, \dots, o_n) , or simply A , with a finite number of primitive operations that generate by composition all functions in A^{A^m} for each $m < \infty$. Such algebras are called *primal*. For example, the algebra $(\{0, 1\}, \wedge, \sim)$ of truth-values in 2-valued logic and, more generally, $(\{0, \dots, n-1\}, \min\{x, y\}, x+1 \pmod{n})$ in n -valued Post logics are primal algebras. The truth-values of the Łukasiewicz-Tarski logics $(\{0, \dots, n-1\}, C, N)$, where $Cxy = \max\{0, y-x\}$, $Nx = n-1-x$, do not form a primal algebra, but if the 0-ary (constant) operation 1 is admitted, then $(\{0, \dots, n-1\}, C, N, 1)$ becomes a primal algebra. Note that any primal algebra is finite, for if it were infinite, then the set of functions would be uncountable, while the set of generated operations would be countable at most.

If O_1, \dots, O_n are the operations symbols in the language for the operations o_1, \dots, o_n of a fixed *species (or similarity type)*, then by the *absolutely-free algebra* (of the given species), $(\Phi_k, o_1, \dots, o_n)$, with k generators x_1, \dots, x_k , is meant the set of all formal expressions defined inductively as follows:

1°. $x_1, \dots, x_k \in \Phi_k$;

2°. for each $i = 1, \dots, n$, if $\phi_1, \dots, \phi_{k_i} \in \Phi_k$, then also

$O_i(\phi_1, \dots, \phi_{k_i}) \in \Phi_k$;

with the operations defined by setting

$$o_i(\phi_1, \dots, \phi_{k_i}) = O_i(\phi_1, \dots, \phi_{k_i}).$$