CONTINUOUS IMAGES OF BOREL SETS

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1. **Introduction.** It is well known that in a complete separable metric space every Borel set is the one-to-one continuous image of a $\mathcal{G}_\delta$ in some other such space and that the countable-to-one continuous image of a Borel set is Borel.

In a general topological space, if K denotes the family of compact sets, G. Choquet [1] has pointed out that the proper families to consider in this context are Borelian K instead of Borel K and $K_{\sigma\delta}$ instead of $\mathcal{G}_\delta$ (see §2 for definitions). Let a space have property I if it is Hausdorff and the difference of two compact sets is a K_σ. It has been shown by Choquet [1] that if Y has property I then every Borelian K set in Y is the one-to-one continuous image of a $K_{\sigma\delta}$ in some compact Hausdorff X. On the other hand, in a previous paper [2] we proved that if X has property I then the countable-to-one continuous image of a $K_{\sigma\delta}$ in X to a compact Hausdorff space Y is Borelian K in Y.

In this paper we complete the picture. We first show that the difference of two compact sets is a K_σ iff it is analytic and conclude that a space has property I iff it is Hausdorff and Borel K = Borelian K, thereby answering a question raised by Choquet [1, p. 139]. We then prove that if X has property I then every Borel K set in X is the one-to-one continuous image of a $K_{\sigma\delta}$ in some Y, where Y also has property I. Making use of our previous result, we conclude that the countable-to-one continuous image of a Borel K set in X to a compact Hausdorff space Y is Borelian K and it too has property I.

We are unable to determine whether the condition that X have property I can be eliminated from the hypotheses, i.e., whether in any compact Hausdorff space X, every Borelian K set is the one-to-one continuous image of a $K_{\sigma\delta}$ in some other compact Hausdorff space and whether the countable-to-one (or even one-to-one) image of a Borelian K (or even a $K_{\sigma\delta}$) set in X into a compact Hausdorff space is also Borelian K.

2. **Notation and basic definitions.**

2.1. ω denotes the set of all non-negative integers.

2.2. $K(X)$ is the family of all compact sets in X.

2.3. $K_\sigma(X) = \{A : A = \bigcup_{i\in\omega} B_i$ for some sequence B with $B_i \in K(X)$ for $i\in\omega\}$.

2.4. $K_{\sigma\delta}(X) = \{A : A = \bigcap_{i\in\omega} B_i$ for some sequence B with $B_i \in K_\sigma(X)$ for $i\in\omega\}$.

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2.5. Borel $K(X)$ is the smallest family H such that $K(X) \subset H$ and if $A_i \in H$, for $i \in \omega$, then $\bigcup_{i\in\omega} A_i \in H$ and $A_0 - A_1 \in H$.

2.6. Borelian $K(X)$ is the smallest family H such that $K(X) \subset H$ and if $A_i \in H$, for $i \in \omega$, then $\bigcup_{i\in\omega} A_i \in H$ and $\bigcap_{i\in\omega} A_i \in H$.

2.7. A is analytic in X iff A is the continuous image of a $K_{\sigma\delta}(X')$ for some Hausdorff space X'.

2.8. X has property I iff X is Hausdorff and, for every A and B in $K(X)$, $A - B \in K_\sigma(X)$.

2.9. X has a countable compact base iff X is Hausdorff and there exists a sequence C with $C_i \in K(X)$, for $i \in \omega$, such that if A is open in X and $x \in A$ then, for some $i \in \omega$, $x \in C_i \subset A$.

2.10. $\prod_{i\in\omega} Y_i$ denotes the cartesian product of the Y_i, for $i \in \omega$.

2.11. The union topology for $\bigcup_{i\in\omega} Y_i$, where $Y_i \cap Y_j = 0$ for $i \neq j$, is the topology in which A is open iff $A = \bigcup_{i\in\omega} \alpha_i$ where α_i is open in Y_i for $i \in \omega$.

3. **Property I and countable compact bases.** In this section, we study conditions under which a set has property I or a countable compact base. The main results are Theorems 3.1, 3.3, 3.5. The other results are needed in the next section.

3.1. THEOREM. *Let X be Hausdorff, A and B in $K(X)$. Then $A - B$ is analytic in X iff $A - B \in K_\sigma(X)$.*

PROOF. If $A - B \in K_\sigma(X)$ then clearly $A - B$ is analytic in X. Suppose now that $A - B$ is analytic in X. Then by Theorem 2.3 in [**3**], $A - B$ is Lindelöf in X, i.e., any open covering of $A - B$ can be reduced to a countable subcovering. Let

$$G = \{\beta : \beta \text{ is open in } X \text{ and } B \subset \beta\},$$
$$F = \{\alpha : \alpha = X - \text{closure } \beta \text{ for some } \beta \in G\}.$$

Since X is Hausdorff, F is an open covering of $X - B$ and hence a countable subfamily F' covers $A - B$. Let G' be a countable subfamily of G such that

$$F' = \{\alpha : \alpha = X - \text{closure } \beta \text{ for some } \beta \in G'\}$$

and let

$$H = \{\gamma : \gamma = A - \beta \text{ for some } \beta \in G'\};$$

then H is a countable family of compact sets whose union is $A - B$ so that $A - B \in K_\sigma(X)$.

3.2. LEMMA. *Borelian $K(X) \subset$ Borel $K(X)$.*

3.3. THEOREM. *X has property* I *iff X is Hausdorff and Borelian $K(X) = $Borel $K(X)$.*

PROOF. Suppose X has property I. Let H be a maximal family such that $K(X) \subset H$ and if A and B are in H then A and $A-B$ are Borelian $K(X)$. Then we easily check that H is closed under countable unions and intersections so that $H=$Borelian $K(X)$. Thus, if A and B are Borelian $K(X)$ then $A-B$ is also Borelian $K(X)$. Therefore Borel $K(X) \subset$Borelian $K(X)$ and in view of 3.2, Borel $K(X)=$Borelian $K(X)$.

Next, suppose X is Hausdorff and Borelian $K(X)=$Borel $K(X)$. If A and B are in $K(X)$ then $A-B$ is Borelian $K(X)$ and hence (see e.g. [1, p. 142]) $A-B$ is analytic in X. Therefore, by 3.1, $A-B \in K_\sigma(X)$.

3.4. LEMMA. *X has property* I *iff X is Hausdorff and for every $A \in K(X)$ and B open in X we have $A \cap B \in K_\sigma(X)$.*

3.5. THEOREM. *If Y has a countable compact base and X has property* I *then $X \times Y$ has property* I.

PROOF. Let C be a sequence of compact sets in Y such that if U is open in Y and $y \in U$ then, for some $i \in \omega$, $y \in C_i \subset U$. Suppose A is compact and B is open in $X \times Y$. Let, for each $i \in \omega$,

$$\beta_i = \{x\colon \{x\} \times C_i \subset B\}.$$

Then the β_i are open in X. Moreover,

$$B = \bigcup_{i\in\omega} (\beta_i \times C_i)$$

for, if $(x, y) \in B$ then, for some $i \in \omega$,

$$y \in C_i \subset \{z\colon (x, z) \in B\}$$

and hence $x \in \beta_i$ and $(x, y) \in \beta_i \times C_i$. Let α be the projection of A onto X. Then α is compact in X and $\alpha \cap \beta_i \in K_\sigma(X)$ and hence

$$(\alpha \cap \beta_i) \times C_i \in K_\sigma(X \times Y).$$

Since

$$A \cap B = \bigcup_{i\in\omega} (A \cap (\beta_i \times C_i)) = \bigcup_{i\in\omega} (A \cap ((\alpha \cap \beta_i) \times C_i))$$

we see that $A \cap B \in K_\sigma(X \times Y)$. Therefore $X \times Y$ has property I.

3.5a. COROLLARY. *If Y is a metric space and X has property* I *then $X \times Y$ has property* I.

PROOF. If $A \in K(X \times Y)$, let B be the projection of A onto Y. Then B is a compact metric space and hence has a countable compact base and $A \subset X \times B$ so that we may apply 3.5.

3.6. THEOREM. *If X has property* I, $C \in K_{\sigma\delta}(X)$, *$f$ is continuous on C, and $f(C)$ is Hausdorff then $f(C)$ has property* I.

PROOF. Let A be compact and B open in $f(C)$. Then $f^{-1}(A)$ is closed in C and hence $f^{-1}(A) \in K_{\sigma\delta}(X)$. Since $f^{-1}(B)$ is open in C and X has property I, we conclude

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \in K_{\sigma\delta}(X).$$

Thus, $A \cap B$ is the continuous image of a $K_{\sigma\delta}(X)$, i.e., $A \cap B$ is analytic in $f(C)$, and hence, by 3.1, $A \cap B \in K_{\sigma}(f(C))$.

3.7. THEOREM. *Let f be continuous and one-to-one on C and $f(C)$ have property* I. *Then C has property* I.

PROOF. Clearly C must be Hausdorff. If A and B are compact in C, then $f(A)$ and $f(B)$ are compact in $f(C)$ and hence $f(A) - f(B) \in K_{\sigma}(f(C))$. Since f is one-to-one, we have

$$A - B = A \cap f^{-1}(f(A) - f(B)) \in K_{\sigma}(C).$$

3.8. LEMMA. *If, for each $i \in \omega$, Y_i is compact and has a countable compact base, then $\prod_{i \in \omega} Y_i$ is compact and has a countable compact base in the product topology.*

3.9. LEMMA. *If, for each $i \in \omega$, Y_i is compact and has a countable compact base and $Y_i \cap Y_j = 0$ for $i \neq j$ then $\bigcup_{i \in \omega} Y_i$ is locally compact and has a countable compact base in the union topology.*

3.10. LEMMA. *If Y is locally compact and has a countable compact base then its one point compactification has a countable compact base.*

4. Continuous images of Borel sets.

In this section we study one-to-one projections of $K_{\sigma\delta}$ sets and continuous countable-to-one images of Borel sets. The main results are Theorems 4.6 and 4.7.

4.1. LEMMA. *If, for each $i \in \omega$, Y_i is compact and A_i is the one-to-one projection of a $K_{\sigma\delta}(X \times Y_i)$ onto X then $\bigcap_{i \in \omega} A_i$ is the one-to-one projection of a*

$$K_{\sigma\delta}\left(X \times \prod_{i \in \omega} Y_i\right) \qquad \textit{onto } X.$$

PROOF. For each $i \in \omega$, let $C_i \in K_{\sigma\delta}(X \times Y_i)$ and A_i be the one-to-one projection of C_i onto X. For $x \in A_i$, let $h_i(x)$ be the y such that $(x, y) \in C_i$, $Z = \prod_{i\in\omega} Y_i$, and

$$F_i = \{(x, y): x \in A_i, y \in Z \text{ and } y_i = h_i(x)\}.$$

Then F_i is homeomorphic to $C_i \times \prod_{j\in(\omega-\{i\})} Y_j$ so that $F_i \in K_{\sigma\delta}(X \times Z)$. Let

$$D = \bigcap_{i\in\omega} F_i.$$

Then $D \in K_{\sigma\delta}(X \times Z)$ and $(x, y) \in D$ iff $x \in \bigcap_{i\in\omega} A_i$ and $y_i = h_i(x)$ for all $i \in \omega$. Thus, if $(x, y) \in D$ and $(x, y') \in D$ then $y_i = h_i(x) = y_i'$ for all $i \in \omega$ so that $y = y'$ and $\bigcap_{i\in\omega} A_i$ is the one-to-one projection of D onto X.

4.2. LEMMA. *If, for each $i \in \omega$, A_i is the one-to-one projection of a $K_{\sigma\delta}(X \times Y_i)$ onto X, $A_i \cap A_j = 0 = Y_i \cap Y_j$ for $i \neq j$, and $Y' = \bigcup_{i\in\omega} Y_i$ with the union topology then $\bigcup_{i\in\omega} A_i$ is the one-to-one projection of a $K_{\sigma\delta}(X \times Y')$ onto X.*

PROOF. Let $C_i \in K_{\sigma\delta}(X \times Y_i)$, A_i be the one-to-one projection of C_i onto X, $D = \bigcup_{i\in\omega} C_i$. Then $\bigcup_{i\in\omega} A_i$ is the one-to-one projection of D. Moreover, D is a $K_{\sigma\delta}(X \times Y')$, for if

$$C_i = \bigcap_{j\in\omega} B(i, j) \qquad \text{with } B(i, j) \in K_\sigma(X \times Y_i)$$

then for $i \neq i'$, $B(i, j) \cap B(i', k) = 0$ for all $j \in \omega$, $k \in \omega$ and hence

$$D = \bigcup_{i\in\omega} \bigcap_{j\in\omega} B(i, j) = \bigcap_{\in\omega} \bigcup_{i\in\omega} B(i, j)$$

and

$$\bigcup_{i\in\omega} (B_{i,j}) \in K_\sigma(X \times Y').$$

4.3. DEFINITION. A is a special set of uniqueness in X iff there exist Y and C such that Y is compact and has a countable compact base and $C \in K_{\sigma\delta}(X \times Y)$ and A is the one-to-one projection of C onto X.

4.4. THEOREM. *If for each $i \in \omega$, A_i is a special set of uniqueness in X then $\bigcap_{i\in\omega} A_i$ is a special set of uniqueness in X and if $A_i \cap A_j = 0$ for $i \neq j$ then $\bigcup_{i\in\omega} A_i$ is a special set of uniqueness in X.*

PROOF. $\bigcap_{i\in\omega} A_i$ is a special set of uniqueness in X in view of 4.1 and 3.8. To see that if $A_i \cap A_j = 0$ for $i \neq j$ then $\bigcup_{i\in\omega} A_i$ is a special set of uniqueness in X, let A_i be the one-to-one projection of C_i where $C_i \in K_{\sigma\delta}(X \times Y_i)$ and Y_i is compact and has a countable compact base. We may assume that $Y_i \cap Y_j = 0$ for $i \neq j$ for otherwise we may replace Y_i by $Y_i \times \{i\}$. Let $Y' = \bigcup_{i\in\omega} Y_i$ with the union topology and Z be the one-point compactification of Y'. Then by Lemmas 3.9 and 3.10, Z is compact and has a countable compact base. Moreover $K_{\sigma\delta}(X \times Y') \subset K_{\sigma\delta}(X \times Z)$. Hence by 4.2, $\bigcup_{i\in\omega} A_i$ is a special set of uniqueness in X.

4.5. THEOREM. *If X has property* I *and $A \in$ Borel $K(X)$ then A is a special set of uniqueness in X.*

PROOF. If $A \in K_{\sigma\delta}(X)$ then clearly A is a special set of uniqueness in X since we can take $Y = \{0\}$ and A is the projection of $A \times \{0\} \in K_{\sigma\delta}(X \times Y)$. Let H be a maximal family such that $K(X) \subset H$ and if A and B are in H then A and $A - B$ are special sets of uniqueness in X. We shall show that H is closed under countable unions and difference of two sets so that Borel $K(X) \subset H$. Let $A_i \in H$, for $i \in \omega$, and

$$S = \{A : A \text{ is a special set of uniqueness in } X\}.$$

We now check, using 4.4:

(i) $A_0 - A_1 \in H$, for, $A_0 - A_1 \in S$ and for any $B \in H$,

$$(A_0 - A_1) - B = (A_0 - A_1) \cap (A_0 - B) \in S$$

and

$$B - (A_0 - A_1) = (B - A_0) \cup (B \cap A_0 \cap A_1) \in S;$$

(ii) $A_0 \cup A_1 \in H$ and hence

$$\bigcup_{i=0}^{n} A_i \in H \qquad \text{for } n \in \omega$$

for,

$$A_0 \cup A_1 = A_0 \cup (A_1 - A_0) \in S$$

and for any $B \in H$,

$$(A_0 \cup A_1) - B = (A_0 - B) \cup ((A_1 - B) \cap (A_1 - A_0)) \in S,$$
$$B - (A_0 \cup A_1) = (B - A_0) \cap (B - A_1) \in S;$$

(iii) $\bigcup_{i\in\omega} A_i \in H$, for, let

$$A_n' = A_n - \bigcup_{i=0}^{n-1} A_i.$$

Then by (i) and (ii), $A_n' \in H$ and $A_n' \cap A_k' = 0$ for $n \neq k$. Hence

$$\bigcup_{i\in\omega} A_i = \bigcup_{n\in\omega} A_n' \in S$$

and for any $B \in H$,

$$\bigcup_{i\in\omega} A_i - B = \bigcup_{n\in\omega} (A_n' - B) \in S,$$

$$B - \bigcup_{i\in\omega} A_i = \bigcap_{i\in\omega} (B - A_i) \in S.$$

4.6. THEOREM. *Let X have property* I. *Then A is Borel $K(X)$ iff, for some $B \in K_\sigma(X)$, $A \subset B$ and there exist X', C, f such that X' has property* I, *$C \in K_{\sigma\delta}(X')$, f is continuous and one-to-one on C and $A = f(C)$.*

PROOF. If $A \in$ Borel $K(X)$ then by 4.5 A is the one-to-one projection of a $K_{\sigma\delta}(X \times Y)$ for some Y that is compact and has a countable compact base. By 3.5, $X \times Y$ has property I. Moreover, since X has property I, by 3.3 Borel $K(X)$ = Borelian $K(X)$ and hence $A \subset B$ for some $B \in K_\sigma(X)$.

The converse is given by Theorem 6.3 in [2] again with the help of 3.3.

4.7. THEOREM. *If X has property* I, *A is Borel $K(X)$, f is continuous and countable-to-one on A to some Hausdorff space Y, and $Y \in K_\sigma(Y)$ then $f(A)$ is Borelian $K(Y)$ and $f(A)$ has property* I.

PROOF. In view of 4.6, $f(A)$ is the continuous countable-to-one image of a $K_{\sigma\delta}$ (X') for some X' that has property I. Hence by Corollary 6.10 in [2], $f(A)$ is Borelian $K(Y)$. By 3.6, $f(A)$ also has property I.

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