

A NOTE ON ARCS OF FINITE CYCLIC ORDER¹

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1. Definitions and statement of theorem. An arc A in the conformal plane means the locally topological image of the closed unit interval, and small letters p, q , etc. will be used to denote points of A or equivalently corresponding points of the parameter interval. An arc without double points is called a simple arc. If C is a circle and $p \in A \cap C$ is an interior point of A then C is said to support [intersect] A at p if there is a neighborhood of p on A such that the two half-neighborhoods determined by p lie in the same component [different components] of the complement of C .

Arc A is said to have finite cyclic order k if k is the maximum number of common points of A with any circle.

A circle C is called a general tangent circle to arc A at a point $p \in A$ provided there exist triples of distinct points (r_i, r'_i, R_i) with sequences $\{r_i\}$ and $\{r'_i\}$ converging on A to p , $\{R_i\}$ converging to some point, and such that $C = \lim C(r_i, r'_i, R_i)$, where $C(r_i, r'_i, R_i)$ denotes the unique circle through these distinct points.

The definitions and notations above are those of Lane and Scherck [2]. The purpose of this note is to establish the following result.

THEOREM 1.1. *If A is a simple arc of finite cyclic order k and C is a circle meeting A in k points, then (a) C intersects A at any common interior point, and (b) C is not a general tangent circle to A at any common point.*

It is to be noted that the theorem involves no smoothness assumptions on A except as these are implied by the finite cyclic order. It is well known, however, that the hypothesis implies conformal differentiability at the endpoints and hence at any interior point one-sided conformal differentiability with respect to both adjoining arcs [1; 2; 3].

2. Proof of part (a) of the theorem. Let $q_i, i=1, \dots, k$, be the points of $A \cap C$ taken in order on A , and let A' be the subarc of A with endpoints q_1 and q_k . Arc A' clearly also has cyclic order k . Let C be oriented and let α_1 and α_2 denote respectively the directed arcs $q_1 q_k$ and $q_k q_1$ of C . Let $l_j [r_j]$, $j=1, 2$, be the number of points of

Received by the editors July 18, 1960.

¹ This research was done with partial support of a fellowship from the National Science Foundation.

$A \cap \alpha_j$ where C supports A' and at which A' is locally to the left [right] of α_j , and let i be the number of points of intersection of A' and C . Since every point of $A' \cap C$ is an intersection, a point of support, or an endpoint, it follows that

$$(2.1) \quad k = l_1 + l_2 + r_1 + r_2 + i + 2.$$

Consider the pencil of circles of the first kind with fundamental points q_1 and q_k , and let C_1 be a member of this pencil arbitrarily close to C but such that α_1 has been displaced to its right and hence α_2 to its left. Each of the r_1 right-hand supports on α_1 is thus replaced by at least two intersections of A' with C_1 and similarly each of the l_2 left-hand supports on α_2 is replaced by at least two intersections. Each of the i intersections in $A' \cap C$ is replaced by at least one intersection in $A' \cap C_1$ and the endpoints are unaltered. Hence the number of points in $A' \cap C_1$ is at least $2r_1 + 2l_2 + i + 2$. By definition of cyclic order this is at most k , whence from (2.1) it follows that $r_1 + l_2 \leq l_1 + r_2$. A similar argument applied to circle C_2 of the pencil displaced from C in the opposite sense yields the reverse inequality, whence it follows that

$$(2.2) \quad l_1 + r_2 = r_1 + l_2.$$

Consider next the pencil of circles of the second kind containing C and with q_1 as fundamental point, and let C_3 [C_4] be a circle of this pencil arbitrarily close to C but displaced to the left [right] of C . Each of the $l_1 + l_2$ left-hand supports [$r_1 + r_2$ right-hand supports] in $A' \cap C$ is replaced by at least two intersections in $A' \cap C_3$ [$A' \cap C_4$]. For definiteness assume A' lies to the left of C at q_k so this endpoint is replaced by at least one intersection in $A' \cap C_3$. Since q_1 is unaltered the number of points in $A' \cap C_3$ is at least $2l_1 + 2l_2 + i + 2$, and since as before this cannot exceed k it follows by (2.1) that $l_1 + l_2 \leq r_1 + r_2$. A similar argument applied to C_4 shows that $r_1 + r_2 \leq l_1 + l_2 + 1$, the difference being that the endpoint q_k is lost in this displacement. From these inequalities it follows that either (i) $r_1 + r_2 = l_1 + l_2$ or (ii) $r_1 + r_2 = l_1 + l_2 + 1$. But case (ii) is impossible since it would follow that $(r_1 + l_2) + (l_1 + r_2) = 2(l_1 + l_2) + 1$ in which the right-hand side is clearly odd while by (2.2) the left-hand side is even. It follows that

$$(2.3) \quad l_1 + l_2 = r_1 + r_2.$$

Suppose now that $A' \cap C$ actually contains at least one point of support, i.e., that not all of r_1, r_2, l_1, l_2 are zero. Then there is a first such support q as C is traced from q_1 and there exists a point $B \in C$ such that B and q_1 separate q on C from all the other supports. Let

directed arcs q_1B and Bq_1 of C be denoted by \mathfrak{B}_1 and \mathfrak{B}_2 respectively, and for definiteness assume that A' is to the left of C at q . Then $q \in \mathfrak{B}_1$ but all r_1+r_2 of the right-hand supports belong to \mathfrak{B}_2 . Consider the pencil of circles of the first kind with q_1 and B as fundamental points, and let C' be a circle of this pencil arbitrarily close to C but such that \mathfrak{B}_1 has been displaced to its left and hence \mathfrak{B}_2 to its right. The support q is thus replaced by at least two intersections of $A' \cap C'$ and the same is true of each of the r_1+r_2 right-hand supports on \mathfrak{B}_2 . Since q_1 is unchanged and none of the i intersections is lost, the number of points in $A' \cap C'$ is at least $2(r_1+r_2+1)+i+1$. By (2.3) and (2.1) it follows that $2(r_1+r_2+1)+i+1 = r_1+r_2+l_1+l_2+i+3 = k+1 > k$. This is impossible since A' has cyclic order k . Hence $r_1=r_2=l_1=l_2=0$ and $k=i+2$.

If C supports A at one of the endpoints of A' , say q_k , an arbitrarily near circle through q_1 will replace the support at q_k by at least two intersections. Since it still meets A' arbitrarily close to each of its i intersections, the number of points in $C \cap A$ is at least $i+3 = k+1 > k$ which is again impossible. Hence C cannot support A at any common interior point. This establishes part (a) of Theorem 1.1 since A has finite cyclic order and hence C supports or intersects at any common interior point.

3. Proof of part (b) of the theorem. In the notation of §1, if $C = \lim C(r_i, r'_i, R_i)$ where $\{r_i\}$ and $\{r'_i\}$ converge on A to p and $\{R_i\}$ converges, it follows that for any point $R \in C$ where $R \neq p$ $C = \lim C(r_i, r'_i, R)$. For if $R \in C$ the sequence $\{R_i\}$ can be chosen, without altering the sequence of circles, so that $\{R_i\}$ converges to R . But since $R \neq p$ and $\{R_i\}$ converges to R , the angle between $C(r_i, r'_i, R_i)$ and $C(r_i, r'_i, R)$ approaches zero (cf. [2, §3.34]). Hence any limit circle of $C(r_i, r'_i, R)$ contains p and R and is tangent to C , whence it coincides with C and $C = \lim C(r_i, r'_i, R)$ as claimed.

As in §2 let q_1, \dots, q_k be the common points of simple arc A with circle C . Suppose that at one of these points, say q_j , C is a general tangent circle to A . By definition there exist sequences $\{r_i\}$ and $\{r'_i\}$ converging to q_j on A and convergent sequence $\{R_i\}$ so that $C = \lim C(r_i, r'_i, R_i)$.

Consider first the case when q_j is one of the points q_1 or q_k , and let q denote the other of these points. Taking q as the point R above, it follows that $C = \lim C(r_i, r'_i, q)$. By §2 each of the $k-2$ points q_2, \dots, q_{k-1} is an intersection of A and C whence, for large enough i , $A \cap C(r_i, r'_i, q)$ has an intersection arbitrarily close to each of these $k-2$ points. Since, in addition, $A \cap C(r_i, r'_i, q)$ contains the three

points r_i, r'_i, q , it contains at least $k+1$ points which is impossible since A has cyclic order k . Hence C cannot be a general tangent circle at q_1 or q_k .

Consider next the case when q_j is one of the points q_2, \dots, q_{k-1} , and let q be either q_1 or q_k . As before it follows that $C = \lim C(r_i, r'_i, q)$ and that, for sufficiently large i , $A \cap C(r_i, r'_i, q)$ has at least one intersection arbitrarily close to each of the $k-2$ points q_2, \dots, q_{k-1} , one of which is q_j . Since $A \cap C(r_i, r'_i, q)$ contains at least two points arbitrarily near q_j , namely r_i and r'_i , and also contains q , it contains at least k points. By §2 it follows that all the points are intersections except perhaps q , which may be an endpoint. Thus r_i and r'_i are both intersections. But since C intersects A at q_j , then for large enough i the number of intersections of A with $C(r_i, r'_i, q)$ in an arbitrarily small neighborhood of q_j must be odd. Thus there is another common point of $A \cap C(r_i, r'_i, q)$ near q_j and hence at least $k+1$ points in all, which is a contradiction as before. This completes the proof of the theorem.

4. Consequences. Two simple consequences of Theorem 1.1 may be written at once.

COROLLARY 4.1. *If a simple closed curve of finite cyclic order k is met in k points by a circle C , then C intersects the curve at every common point and is not a general tangent circle to it at any of them.*

This is obtained at once from Theorem 1.1 by considering a subarc A of the simple closed curve which contains as interior points all the common points with C .

COROLLARY 4.2. *The cyclic order of a simple arc A is the same as that of the open subarc obtained by deleting the endpoints.*

The case of infinite cyclic order is trivial since this cannot be changed by dropping any finite number of points. If k is the finite cyclic order of A and circle C meets A in the k points q_1, \dots, q_k it follows from part (a) of the theorem that all interior common points are intersections whence any circle C' arbitrarily close to C will intersect A close to each of the $k-2$ points q_2, \dots, q_{k-1} . If C' is chosen to meet the open arc q_1q_k of A arbitrarily close to q_1 and to q_k it will meet this open arc k times. It follows that the arc consisting of the interior points of A has at least cyclic order k . But since the cyclic order cannot exceed that of A the cyclic order is exactly k , as was to be shown.

The special case of this result relating to arcs of cyclic order 3 is

discussed by Lane and Scherk [2, §3.3] in developing the properties of such arcs.

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