ON ORDER-CONVERGENCE

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1. **Introduction.** Let X be a set partially ordered by a relation $\leqq$ and possessing least and greatest elements O and I, respectively. Let $\{f(\alpha), \alpha \in D\}$ be a net on the directed set D with values in X (our terminology and notation for nets are those of Kelley [**4**]). A number of authors have attached various meanings (many of them distinct) to the statement "f order-converges to the element y." We shall discuss two of these notions of convergence which, although distinct, are intimately related. The first, which we shall call "o-convergence," is due in essence to Birkhoff [**1**] and has been studied by Frink [**3**] and McShane [**5**]. The second was introduced by Rennie [**6**; **7**], and was employed by Ward [**8**] (using the terminology of filters). Following Rennie's notation, we shall call this second type of convergence "o_2-convergence." It is natural to ask the question: in what class of partially ordered sets are these two notions of convergence equivalent? Although a theorem characterizing such partially ordered sets would be excessively involved, we shall show that it is possible to obtain a convenient condition on the partially ordered set X which is necessary and sufficient for the associated concepts of "lim inf" (and dually of "lim sup") to be equivalent. (For practical purposes this might be considered as an approximate solution of the problem.) Our condition takes a particularly simple form by making use of the concept of ideal which was introduced by Frink [**2**]. This result, which is our main theorem, is obtained as a consequence of a correspondence which we establish between nets and ideals.

2. **Preliminaries.** We denote set inclusion by $\subseteq$, reserving $\subset$ for proper inclusion. If S is a subset of the partially ordered set X, we say that S is *up-directed* (*down-directed*) if and only if every finite subset of S has an upper bound (lower bound) in S. "Directed" will be used in a general sense to denote either "up-directed" or "down-directed." For any $S \subseteq X$, we write $S^* = \{x \in X \mid x \geqq a \text{ for all } a \in S\}$, and $S^+ = \{x \in X \mid x \leqq a \text{ for all } a \in S\}$. For $(S^*)^+$ we shall write S^{*+}, and dually.

We shall always consider the domain of a net f to be an up-directed partially ordered set. If f is a net on D to X, and $\beta \in D$, we define $E_f(\beta) = \{f(\alpha) \mid \alpha \geqq \beta\}$. We also define

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$$P_f = \bigcup \{ [E_f(\beta)]^+ \mid \beta \in D \},$$
$$Q_f = \bigcup \{ [E_f(\beta)]^* \mid \beta \in D \}.$$

We now give the Birkhoff-Frink-McShane definition of o-convergence.

DEFINITION 1. If $\{f(\alpha), \alpha \in D\}$ is a net in X, we say that f *o-converges to* y (and write $y = o\text{-}\lim f$) if and only if there exist subsets M and N of X such that

(i) M is up-directed and N is down-directed,

(ii) $y = \text{l.u.b. } M = \text{g.l.b. } N$,

(iii) for each $m \in M$ and $n \in N$, there exists $\beta \in D$ such that $m \leqq f(\alpha) \leqq n$ for all $\alpha \geqq \beta$.

REMARK. A condition equivalent to (iii) is

(iii)′ $M \subseteq P_f$ and $N \subseteq Q_f$.

It is clear that (iii) implies (iii)′. Conversely, assume that M and N are sets for which (iii)′ holds and let $m \in M$ and $n \in N$. Then there exist $\alpha_1 \in D$ and $\alpha_2 \in D$ such that $m \in [E_f(\alpha_1)]^+$ and $n \in [E_f(\alpha_2)]^*$. Let β be an element of D with $\beta \geqq \alpha_1$, $\beta \geqq \alpha_2$. Then $E_f(\beta) \subseteq E_f(\alpha_1) \cap E_f(\alpha_2)$, and $m \leqq f(\alpha) \leqq n$ for all $\alpha \geqq \beta$.

It should be noted that "f is o-convergent" does *not* imply "the sets P_f and Q_f are directed." A simple example to illustrate this is given below in §4.

The following definition is that of Rennie and Ward.

DEFINITION 2. If f is a net in X, we write $y = o_2\text{-}\lim\inf f$ if and only if $y = \text{l.u.b. } P_f$; and $y = o_2\text{-}\lim\sup f$ if and only if $y = \text{g.l.b. } Q_f$. If $\text{l.u.b. } P_f = \text{g.l.b. } Q_f = y$, we say that f *o_2-converges to* y (and write $y = o_2\text{-}\lim f$).

We also give another characterization of o_2-convergence.

THEOREM 1. *Let $\{f(\alpha), \alpha \in D\}$ be a net in X. Then $y = o_2\text{-}\lim f$ if and only if there exist subsets M and N of X such that*

(i) $y = \text{l.u.b. } M = \text{g.l.b. } N$, *and*

(ii) *for each $m \in M$ and $n \in N$, there exists $\beta \in D$ such that $m \leqq f(\alpha) \leqq n$ for all $\alpha \geqq \beta$.*

PROOF. If $y = o_2\text{-}\lim f$, one merely takes $M = P_f$, $N = Q_f$, and (i) and (ii) are satisfied. To prove the converse, suppose that f is a net in X for which there exist sets M and N satisfying (i) and (ii). Condition (ii) implies that $M \subseteq P_f$, $N \subseteq Q_f$. Since $M^{*+} = \{x \in X \mid x \leqq y\}$ and $N^{+*} = \{x \in X \mid x \geqq y\}$, we have $M^{*+} \cap N^{+*} = \{y\}$. But $M^{*+} \subseteq P_f^{*+}$, $N^{+*} \subseteq Q_f^{+*}$, and hence $y \in P_f^{*+} \cap Q_f^{+*}$. But we have $Q_f \subseteq P_f^*$, $Q_f^+ \supseteq P_f^{*+}$, and hence $y \in Q_f^+ \cap Q_f^{+*}$. This implies $y = \text{l.u.b. } Q_f^+ = \text{g.l.b. } Q_f$. By the dual argument we also have $y = \text{l.u.b. } P_f$. Hence $y = o_2\text{-}\lim f$.

As an immediate corollary of Theorem 1 we have the following result.

COROLLARY. *If f is a net in a partially ordered set X, then $y=o$-$\lim f$ implies $y=o_2$-$\lim f$.*

The equivalence of conditions (iii) and (iii)′ in Definition 1 suggests the following natural definitions of "lim inf" and "lim sup" for o-convergence. Note that these definitions do not coincide with those of McShane [5, p. 15], which are much more restrictive.

DEFINITION 3. $y=o$-$\lim\inf f$ if and only if $y\in P_f^*$ and there exists an up-directed subset M of P_f with $y=$ l.u.b. M. $y=o$-$\lim\sup f$ if and only if $y\in Q_f^+$ and there exists a down-directed subset N of Q_f with $y=$ g.l.b. N.

THEOREM 2. *If f is a net in a partially ordered set X, then*

(i) *$y=o$-$\lim\inf f$ implies $y=o_2$-$\lim\inf f$ and dually,*

(ii) *$y=o$-$\lim f$ if and only if $y=o$-$\lim\inf f=o$-$\lim\sup f$.*

PROOF. (i) Let $y=o$-$\lim\inf f$. Let M be up-directed, $M\subseteq P_f$, and $y=$ l.u.b. M. Then $M^*\supseteq P_f^*$. Hence $x\in P_f^*$ implies $x\geqq y$. Since $y\in P_f^*$, we have $y=$ l.u.b. P_f.

(ii) $y=o$-$\lim f$ implies $y=o_2$-$\lim f$ (corollary to Theorem 1). Hence $y\in P_f^*$, $y\in Q_f^+$, and the remaining requirements of Definition 3 are trivially satisfied. The converse is also trivial.

3. Nets and ideals.

The following definition is due to Frink [2].

DEFINITION 4. A subset K of a partially ordered set X is an *ideal* (*dual ideal*) in X if and only if for every finite subset F of K we have $F^{*+}\subseteq K$ ($F^{+*}\subseteq K$). An ideal (dual ideal) is *normal* if and only if $K^{*+}=K$ ($K^{+*}=K$).

It is readily verified that the set of all ideals of X, partially ordered by set inclusion, forms a complete lattice.

The following theorem, which gives information about the structure of non-normal ideals, will be of some use to us.

THEOREM 3. *If K is a non-normal ideal in a partially ordered set X, then*

(i) *there exists a chain in K with no upper bound in K, or*

(ii) *K contains an infinite set S of maximal elements such that $x\in K$ implies $x\leqq m$ for some $m\in S$.*

PROOF. Suppose that (i) does not hold: i.e., suppose that every chain in K has an upper bound in K. Then by Zorn's lemma K has a nonempty set S of maximal elements. If $x\in K$, let Z be a maximal chain in K which contains x. By our assumption, Z has an upper bound m in K. Then $x \leqq m$ and $m \in S$ (by maximality of Z). It remains to prove that S is infinite. Since we have shown above that $S^* = K^*$, it follows that $S^{*+} = K^{*+}$. If S were finite, we would then have $S^{*+} = K^{*+} \subseteq K$, since K is an ideal. But this contradicts the hypothesis that K is non-normal.

COROLLARY. *In any partially ordered set, a finite ideal is normal.*

We now prove a theorem which sets up a correspondence between nets and ideals.

THEOREM 4. *A subset K of a partially ordered set X is an ideal (dual ideal) if and only if there exists a net g in X such that $K = P_g$ $(K = Q_g)$.*

PROOF. Let $\{g(\alpha), \alpha \in D\}$ be a net in X, and let $F = \{x_1, \cdots, x_n\}$ be a finite subset of P_g. Then for each $i = 1, \cdots, n$, there exists $\beta_i \in D$ such that $x_i \in [E_g(\beta_i)]^+$. Let β be an element of D such that $\beta \geqq \beta_i$ for all i. Then $E_g(\beta) \subseteq F^*$, and hence $F^{*+} \subseteq [E_g(\beta)]^+ \subseteq P_g$. Hence P_g is an ideal. The obvious dual proof applies to Q_g.

To prove the converse we consider two cases. First, let K be an infinite ideal in X. Let $\mathfrak{F}$ be the family of all finite subsets of K, and let $\mathfrak{F}$ be partially ordered by set inclusion. For each $F \in \mathfrak{F}$, let W_F be an up-directed partially ordered set in 1:1 correspondence with F^*, and containing a least element α_F. This partial order on W_F, which we again denote by $\leqq$, of course need not correspond to the order defined on F^* as a subset of X. For $\alpha \in W_F$, let the corresponding element of F^* be denoted by a_α. Define $D = \{(F, \alpha) \mid F \in \mathfrak{F} \text{ and } \alpha \in W_F\}$. We order D "lexicographically" by defining $(F_1, \alpha_1) < (F_2, \alpha_2)$ if and only if $F_1 \subset F_2$, or, when $F_1 = F_2$, if $\alpha_1 < \alpha_2$. This is a partial order with respect to which D is up-directed. Let g be a net on D to X defined by $g(F, \alpha) = a_\alpha$. We shall prove that $P_g = K$. Let (F_1, α_1) be any element of D. Then, since K is infinite, there exists $F \in \mathfrak{F}$ with $F_1 \subset F$; and hence $E_g(F_1, \alpha_1) \supseteq E_g(F, \alpha_F) = F^*$. Then $[E_g(F_1, \alpha_1)]^+ \subseteq F^{*+} \subseteq K$. Thus $P_g \subseteq K$. To prove the reverse inclusion, let $x_0 \in K$ and let F be the set consisting of the single element x_0. Then $E_g(F, \alpha_F) = \{x \in X \mid x \geqq x_0\}$ and hence $x_0 \in [E_g(F, \alpha_F)]^+ \subseteq P_g$. Hence $P_g = K$.

We assume now that K is a finite ideal, and hence normal, by the corollary to Theorem 3. Let E be a set which is in 1:1 correspondence with K^* and which is up-directed by some partial ordering relation $\leqq$. For $\alpha \in E$, let a_α denote the corresponding element of K^*. Let $D = \{(i, \alpha) \mid i \text{ is a positive integer and } \alpha \in E\}$. We again make D an up-directed set by defining $(i_1, \alpha_1) < (i_2, \alpha_2)$ if and only if $i_1 < i_2$ or, when $i_1 = i_2$, if $\alpha_1 < \alpha_2$. Let g be a net on D to K^* defined by $g(i, \alpha) = a_\alpha$.

From our construction it is clear that $E_g(i, \alpha)=K^*$ for all $(i, \alpha)\in D$, and hence $[E_g(i, \alpha)]^+=K^{*+}=K$ for all (i, α). Thus we again have $P_g=K$.

The following corollary, which gives us a new characterization of a complete lattice, may be of some incidental interest.

COROLLARY. *A partially ordered set X with elements O and I is a complete lattice if and only if o_2-*lim inf *f exists for every net f in X.*

PROOF. It is trivial that lim inf f exists for every net f in a complete lattice. To prove the converse, let $S\subseteq X$ and let K be the smallest ideal in X which contains S. Since $K=P_g$ for some net g in X, it follows from our hypothesis that $y=$l.u.b. K exists. Let m be any element of S^*. Since $\{x \mid x\leqq m\}$ is an ideal containing S, we have $K\subseteq\{x \mid x\leqq m\}$ and hence $m\geqq y$. Then $y=$l.u.b. S, and X is a complete lattice.

For convenience we introduce another definition.

DEFINITION 5. A partially ordered set X *has Property* A if and only if whenever K is a non-normal ideal in X with a least upper bound y in X, there exists $M\subseteq K$ such that M is up-directed and $y=$l.u.b. M.

We now prove our main result. The dual formulatiou may be left to the reader.

THEOREM 5. *A partially ordered set X has Property* A *if and only if for every net f in X and every $y\in X$, $y=o_2$-*lim inf *f is equivalent to $y=o$-*lim inf *f.*

PROOF. Let X have Property A. By Theorem 2, we need only to show that if f is a net in X with $y=o_2$-lim inf f, then $y=o$-lim inf f. If $y\in P_f$, then trivially P_f is up-directed and in Definition 3 we may take $M=P_f$. Suppose, then, that $y\notin P_f$. Then it follows that P_f is a non-normal ideal, since $y=$l.u.b. P_f and $P_f^{*+}=\{x\in X \mid x\leqq y\}\neq P_f$. By hypothesis P_f contains an up-directed subset M with $y=$l.u.b. M, and hence $y=o$-lim inf f.

To prove the converse, suppose that K is a non-normal ideal in X with $y=$l.u.b. K, and suppose that K contains no up-directed subset M with $y=$l.u.b. M. By Theorem 4, there exists a net g in X with $K=P_g$. Then $y=o_2$-lim inf g, but by Definition 3 we cannot have $y=o$-lim inf g.

4. **Some examples.** We first give an example of an o-convergent net f for which P_f is not up-directed. Let A and B be infinite ascending chains $a_1<a_2<\cdots<a_n<\cdots$ and $b_1<b_2<\cdots<b_n<\cdots$, each of order type of the positive integers. Let a_i and b_j be incom-

parable for all i and j. Let $Y=\{y_n | n=1, 2, \cdots\}$ be another sequence of elements with y_i and y_j incomparable for all i and j. Define $y_i > a_j$ if and only if $i \geqq j$, and also $y_i > b_j$ if and only if $i \geqq j$. Adjoin an element I with $x < I$ for all $x \in A \cup B \cup Y$. Let f be the sequence defined by $f(n) = y_n$. Then $P_f = A \cup B$ and $Q_f = \{I\}$. Also, f is o-convergent to I, since in Definition 1 we may take the set $N = \{I\}$ and $M = A$ (or $M = B$). However, P_f is not up-directed.

We now give an example of a partially ordered set X which does not possess Property A. Let $Y = \{y_n\}$ and $Z = \{z_n\}$ be two sequences of elements. Let y_i and y_j be incomparable for all i, j, and let z_i and z_j be incomparable for all i, j. Define $z_i < y_j$ if and only if $i \leqq j$. Adjoin elements O and I which are upper and lower bounds, respectively, of $Y \cup Z$. Let $X = Y \cup Z \cup \{O\} \cup \{I\}$, and let $K = Z \cup \{O\}$. The reader may verify that K is a non-normal ideal in X with $I =$ l.u.b. K. However, there is no up-directed subset M of K with $I =$ l.u.b. M. Furthermore, if we let $f(n) = y_n$, then $P_f = K$ and $Q_f = \{I\}$. Hence the sequence f is o_2-convergent to I, but $I \neq o$-lim inf f.

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