A BANACH SPACE CHARACTERIZATION OF PURELY ATOMIC MEASURE SPACES

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It is well known [**4**, p. 265; **3**] that the space $L_1[0, 1]$ is not isomorphic with a conjugate space. At the other extreme, it is also well known that l_1 is *isometric* with the conjugate space of c_0. Each of these is an example of a space of all real-valued integrable functions over a measure space (T, μ), a major difference between them being that the measure space underlying $L_1[0, 1]$ has no atoms, while that underlying l_1 is purely atomic. It is natural to conjecture that a space $L_1(T, \mu)$ is isomorphic with a conjugate space if and only if (T, μ) is purely atomic; we will show that this conjecture is false, although it is true for separable L_1 spaces. We prove this result, together with one of our characterizations of purely atomic (T, μ), by using the notion of differentiability of vector-valued functions of bounded variation on $[0, 1]$. (This was the method employed by Gelfand [**4**] in proving the result cited above.) A related result is given in terms of locally uniformly convex spaces [**8**].

Let (T, μ) be a measure space. (We do not assume that T is measurable.) An *atom* $A \subset T$ is a measurable set such that $0 < \mu(A) < \infty$, and for each measurable set $B \subset A$, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. We will consider two atoms to be the "same" if they differ by a set of measure zero. A set S of positive finite measure is *purely atomic* if the set $S \sim \bigcup \{A \subset S: A \text{ is an atom}\}$ has measure zero. (Since atoms are essentially disjoint, μ is countably additive, and $\mu(S) < \infty$, S can contain at most countably many atoms, and hence the above set is measurable.) We say that *the measure space* (T, μ) *is purely atomic* if every subset $S \subset T$ of positive finite measure is purely atomic. We denote by $\mathfrak{A}$ the collection of all atoms $A \subset T$. There are doubtless other possible definitions of "purely atomic"; that the one given here is reasonable is shown by the following lemma.

LEMMA. *If* (T, μ) *is purely atomic, then* $L_1(T, \mu)$ *is isometric with* $l(\mathfrak{A})$ (*and hence is a conjugate space*).

PROOF. The space $l(\mathfrak{A})$ is the set of all real functions y on $\mathfrak{A}$ such that $\|y\| = \sum_{\mathfrak{A}} |y(A)|$ is finite, the summation being taken over the directed system of all finite subsets of $\mathfrak{A}$. (See [**2**] for a proof that $l_1(\mathfrak{A})$ is isometric with the conjugate space of $c_0(\mathfrak{A})$.) We first show

Received by the editors May 9, 1960.

[1] National Science Foundation Postdoctoral Fellow.

that any σ-finite subset S of R is purely atomic, i.e., if $S=\bigcup_{i=1}^{\infty} S_i$, where the S_i are pairwise disjoint sets of positive finite measure, then (letting U be the union of all the atoms in S) we have $\mu(S\sim U)=0$. Indeed, each atom of S is contained in some S_i; since each S_i contains at most countably many atoms, the same is true of S and therefore $S\sim U$ is measurable. If $S\sim U$ were to have positive measure, the equality $S\sim U=\bigcup(S\sim U)\cap S_i$ would imply that at least one set $(S\sim U)\cap S_i$ would have positive measure, and would therefore contain an atom A. Since A would also be an atom in S, but not in U, this would be a contradiction.

Now, if $x\in L_1(T, \mu)$ and A is an atom, then x is constant a.e. on A. Let $(\phi x)(A)=x(A)\mu(A)$; then $\sum_{\mathfrak{A}}|(\phi x)(A)| = \sum |x(A)|\mu(A) \leqq \int_T |x|\,d\mu < \infty$, so $\phi x\in l(\mathfrak{A})$. If $y\in l_1(\mathfrak{A})$, then the element x which equals $y(A)\mu(A)^{-1}$ on each atom and is zero elsewhere is in $L_1(T, \mu)$ and hence ϕ is onto. Since ϕ is clearly linear, we need only show that it is an isometry, i.e. that for each $x\in L_1(T, \mu)$, $\sum_{\mathfrak{A}} |x(A)|\mu(A) = \int_T |x|\,d\mu$.

Let $S(x)=\{t\in T: x(t)\neq 0\}$; this set is easily seen to be σ-finite and therefore purely atomic. Now (letting $U(x)$ be the union of all the atoms in $S(x)$), $\int_T|x|\,d\mu=\sum_{\mathfrak{A}} |x(A)|\mu(A)+\int_{S(x)\sim U(x)}|x|\,d\mu$; since $S(x)$ is purely atomic, the second term is zero and ϕ is an isometry.

A Banach space E is *isomorphic* with a Banach space F if there exists a continuous, linear one-to-one map of E onto F which has a continuous inverse. The existence of a Banach space isomorphic to E is equivalent to the existence of positive constants k and K and norms $\|\cdots\|$ and $|||\cdots|||$ on E such that $k\|x\|\leqq|||x|||\leqq K\|x\|$ for all $x\in E$.

A normed space E is *locally uniformly convex* if for each $x\in E$ such that $\|x\|=1$, and for each $\epsilon>0$, there exists $\delta(x, \epsilon)>0$ such that $\|x+y\|\leqq 2-\delta$ whenever $\|x-y\|\geqq\epsilon$. It is easily seen that uniform convexity [2] implies local uniform convexity, and the latter implies strict convexity; Lovaglia [8] shows that neither of these implications may be reversed. An equivalent formulation in terms of sequences ($\|x\|=1=\|y_n\|$ and $\|x+y_n\|\to 2$ imply $\|x-y_n\|\to 0$) shows that E is locally uniformly convex if and only if each separable subspace of E is locally uniformly convex.

The following theorem has been proved by Lovaglia [8, Theorem 3.1] in a more general context, but in a slightly different way. The adaptation of his proof to this special case is shorter; more importantly, our method of renorming l_1 will enable us to apply the result to nonseparable l_1 spaces.

LOVAGLIA'S THEOREM. *The space l_1 is isomorphic with a locally uniformly convex space.*

PROOF. By l_1 we mean, of course, the space of all sequences x such that $\|x\| = \sum |x_i| < \infty$. Define a new norm of l_1 by

$$\|x\|_1 = (\|x\|^2 + \sum x_i^2)^{1/2};$$

it is easily checked that $\|x\| \leqq \|x\|_1 \leqq (2)^{1/2}\|x\|$. To see that this norm makes l_1 locally uniformly convex, suppose that $\|x\|_1 = 1 = \|y^n\|_1$ and $\|x+y^n\|_1 \to 2$, but $\lim \|x-y^n\|_1$ (and hence $\lim \|x-y^n\|) \neq 0$. Then there exists a subsequence of the y's (say $\{y^n\}$) and $t > 0$ such that $\|x-y^n\| \geqq t$. Since $\|y^n\| \leqq 1$ and $|y_i^n| \leqq 1$ for each i, we can use the diagonal process to obtain a subsequence such that $\|y^n\| \to a$, say, while $y_i^n \to a_i$ for each i. Thus,

$$\lim \sum_{k+1}^{\infty} (y_i^n)^2 = \lim\left(1 - \sum_{i}^{k} (y_i^n)^2 - \|y^n\|^2\right) = 1 - \sum_{i}^{k} a_i^2 - a^2 \equiv b_k^2,$$

say. Now, for each $k \geqq 1$ we have

$$\begin{aligned}
&\|x + y^n\|_1 \\
&\quad \leqq \left\{ \sum_{i}^{k} (x_i + y_i^n)^2 + (\|x\| + \|y^n\|)^2 + \left[\sum_{k+1}^{\infty} (x_i + y_i^n)^2 \right]^{(1/2)2} \right\}^{1/2} \\
&\quad \leqq \left\{ \sum_{i}^{k} (x_i + y_i^n)^2 + (\|x\| + \|y^n\|)^2 \right. \\
&\qquad\qquad \left. + \left[\left(\sum_{k+1}^{\infty} x_i^2 \right)^{1/2} + \left(\sum_{k+1}^{\infty} (y_i^n)^2 \right)^{1/2} \right]^2 \right\}^{1/2},
\end{aligned}$$

the latter being obtained by Minkowski's inequality. Taking limits as $n \to \infty$ and applying the Minkowski inequality once again yields

$$\begin{aligned}
2 &= \lim \|x + y^n\|_1 \\
&\leqq \left\{ \sum_{1}^{k} (x_i + a_i)^2 + (\|x\| + a)^2 + \left[\left(\sum_{k+1}^{\infty} x_i^2 \right)^{1/2} + b_k \right]^2 \right\}^{1/2} \\
&\leqq \left[\sum_{1}^{k} x_i^2 + \|x\|^2 + \sum_{k+1}^{\infty} x_i^2 \right]^{1/2} + \left[\sum_{1}^{k} a_i^2 + a^2 + b_k^2 \right]^{1/2} = 2.
\end{aligned}$$

Since equality holds throughout (for all $k \geqq 1$), it follows that $\|x\| = a$ and $x_i = a_i$ for all i. We see, then, that $t \leqq \|x - y^n\| \leqq \sum_1^k |x_i - y_i^n| + \sum_{k+1}^{\infty} |y_i^n|$, so for $k \geqq 1$,

$$\liminf_{n\to\infty} \sum_{k+1}^{\infty} |y_i^n| \geqq t - \sum_{k+1}^{\infty} |x_i| .$$

If we choose $k \geqq k_0$, say, the right side will be no less than $u > 0$; if k_0 is sufficiently large, we will have $0 < \sum_1^k |x_i| < 1$ and hence, for $k \geqq k_0$,

$$\liminf \sum_{k+1}^{\infty} |y_i^n| \left(\sum_1^k |x_i| \right)^{-1} \geqq u \left(\sum_1^k |x_i| \right)^{-1} > u > 0.$$

Choosing $K \geqq k_0$ such that $(1+u) \sum_1^K |x_i| > \|x\|$, we see that there exists a subsequence of the y's such that

$$\sum_{K+1}^{\infty} |y_i^n| \left(\sum_1^K |x_i| \right)^{-1} > u > 0.$$

Finally, then, we have

$$\sum_1^K |y_i^n| \left(\sum_1^K |y_i^n| \right)^{-1} + \sum_{K+1}^{\infty} |y_i^n| \left(\sum_1^K |x_i| \right)^{-1} > 1 + u$$

so that $\liminf \|y^n\| (\sum_1^K |x_i|)^{-1} \geqq 1 + u$. Since $\liminf \|y^n\| = \lim \|y^n\| = \|x\|$, this contradicts the inequality used in defining K, and the proof is complete.

A function ϕ defined on $[0, 1]$ whose range lies in a normed space E is of *bounded variation* if $\sup \sum \|\phi(r_{i+1}) - \phi(r_i)\| < \infty$, where the supremum is taken over all partitions $0 = r_0 < r_1 < \cdots < r_n = 1$ of $[0, 1]$. We say that ϕ is *differentiable a.e.* if the limit

$$\phi'(r) = \lim_{h\to 0} [\phi(r + h) - \phi(r)] h^{-1}$$

exists for all r in $[0, 1]$ outside a set of Lebesgue measure zero. The space E has *property* (D) if every ϕ from $[0, 1]$ into E which is of bounded variation is differentiable a.e. Note that E has property (D) if and only if E has property (D) under an equivalent norm, i.e. property (D) is "preserved" under isomorphism.

We now state a theorem concerning property (D) which will be of use in what follows.

GELFAND'S THEOREM. *If a separable Banach space E is isomorphic with a conjugate space, then E has property* (D).

Gelfand proved this in [**4**, p. 264]; an interesting proof has also been given by Alaoglu in [**1**].

THEOREM. *Let (T, μ) be any measure space; then assertions* (i) *and* (iii) *are equivalent and imply* (ii):

(i) (T, μ) *is purely atomic.*

(ii) $L_1(T, \mu)$ *is isomorphic with a locally uniformly convex Banach space.*

(iii) $L_1(T, \mu)$ *has property* (D).

PROOF. (i) implies (ii). If (T, μ) is purely atomic, we may assume. by virtue of the above lemma, that $L_1(T, \mu)$ is $l_1(S)$ for some set S. Define a new norm on l_1 by $\|x\|_1^2 = (\sum |x(s)|)^2 + \sum x(s)^2$; then $\sum |x(s)| \leqq \|x\|_1 \leqq (2)^{1/2} \sum |x(s)|$. We need only show that, under this norm, every separable subspace (and hence $l_1(S)$ itself) is locally uniformly convex. Let M be any separable subspace of $l_1(S)$ and let $\{x_n\}_1^\infty$ be a dense sequence in M. As in the proof of the lemma, the support $S(x_n)$ of each x_n is countable, so the set $S_M = \bigcup_1^\infty S(x_n)$ is also countable. Since $S(x) \subset S_M$ for each $x \in M$, we see that $M \subset l_1(S_M)$, the (separable) subspace of all elements in $l_1(S)$ which vanish outside S_M. But, by our proof of Lovaglia's theorem, $l_1(S_M)$ is locally uniformly convex under the norm induced by $\| \cdot \cdot \|_1$.

(iii) implies (i). Suppose that (T, μ) is *not* purely atomic. Then T contains a subset S' of finite positive measure which is not the union of atoms; letting $S = S' \sim \bigcup \{A : A \text{ is an atom}, A \subset S'\}$, we see that $0 < \mu(S) < \infty$ and S contains no atoms. In the terminology of [**5**], the restriction of μ to the measurable subsets of S is a convex measure, i.e. there exists a measurable function f defined on S with range $[0, 1[$ such that $\mu\{s \in S : f(s) < r\} = r\mu(S)$ for each $0 \leqq r \leqq 1$. Define ϕ on $[0, 1]$ by letting $\phi(r)$ be the characteristic function of $\{s \in S : f(s) < r\}$. Then $\phi(r) \in L_1(T, \mu)$ and $\|\phi(r) - \phi(r')\| = |r - r'| \mu(S)$, so ϕ is of bounded variation. It is not differentiable at any point of $]0, 1[$, however, since if $0 < r < 1$, let $0 < h < \min(r, 1 - r)$ and verify that $\| [\phi(r + h) - \phi(r)] h^{-1} - [\phi(r - h) - \phi(r)](-h)^{-1} \| = 2\mu(S)$. Thus, $L_1(T, \mu)$ does not have property (D), and the proof is complete.

(i) implies (ii). We simply observe that a function of bounded variation has at most countably many discontinuities, so that its range lies in a separable subspace M of $l_1(S)$. Since $M \subset l_1(S_M)$ and the latter is a separable conjugate space, Gelfand's theorem applies.

COROLLARY. *Suppose that* $L_1(T, \mu)$ *is separable. Then* (T, μ) *is purely atomic if and only if* $L_1(T, \mu)$ *is isomorphic with a conjugate space.*

PROOF. By the lemma, we see that if (T, μ) is purely atomic, then $L_1(T, \mu)$ is *isometric* with a conjugate space. The Gelfand theorem and "(iii) implies (i)" of the above theorem prove the converse.

By a result of Kakutani [**6**], the second conjugate E of $L_1[0, 1]$ is an abstract (L)-space and hence [**7**] is of the form $L_1(T, \mu)$ for some measure space (T, μ). Since Lebesgue measure on $[0, 1]$ is nonatomic, our theorem shows that $L_1[0, 1]$ does not have property (D), and hence (using the natural embedding of a Banach space into its second conjugate) E does not have property (D). By the theorem again, E is not purely atomic, i.e. *there exists a measure space* (T, μ), *which is not purely atomic, such that* $L_1(T, \mu)$ *is a conjugate space.*

The problem posed by Dieudonné in [3] remains open: Characterize those (T, μ) for which $L_1(T, \mu)$ is isometric (or isomorphic) with a conjugate space.

Added in proof. M. I. Kadec [Izvestia Vyših Učebnyh Zavedenii. Mat. vol. 6 (13) (1959) pp. 51–57] has proved the interesting fact that every separable Banach space is isomorphic with a locally uniformly convex space.

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