ITERATES OF CONDITIONAL EXPECTATION OPERATORS

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1. **Introduction.** Let $\{T_n\}$ be a sequence of conditional expectation operators in $L_1 = L_1(W, F, P)$ where (W, F, P) is a probability space. Let $S_n = T_n \cdots T_2 T_1$. It is known [1, p. 331] that if $\{T_n\}$ is monotone decreasing, that is, if the range of T_{n+1} is a subset of the range of T_n for all n, then for each x in L_1 the sequence $\{S_n x\}$ converges almost everywhere. Here, the pointwise convergence behavior of $\{S_n x\}$ is studied under other conditions. For example, if $T_{2n-1} = T_1$ and $T_{2n} = T_2$ for all n, does $\{S_n x\}$ converge almost everywhere? This question was first posed by J. L. Doob. It is proved here that if x is in L_2, then this is indeed the case, and, furthermore, $\sup_n |S_n x|$ is in L_2. Several of the preliminary results needed, especially Theorems 1 and 2, seem to be of some interest in their own right. The linear spaces mentioned in this paper may be either real or complex. All of our results hold with either interpretation.

2. **Self-adjointness as a Tauberian condition for almost everywhere convergence.**

LEMMA 1. *If $\{a_n\}$ is a convex and bounded real number sequence, then*

$$\sum_{n=1}^{\infty} n\Delta^2 a_n = a_1 - \lim_{n\to\infty} a_n.$$

PROOF. Write the series in the form $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \Delta^2 a_n$, invoke Fubini, and sum.

LEMMA 2. *Suppose that T is a linear self-adjoint operator with norm $|T| \leqq 1$ in a complete inner product space H. Then, for each x in H,*

$$\sum_{n=1}^{\infty} n\|T^n x - T^{n+2}x\|^2 \leqq \|x\|^2, \tag{1}$$

the strong limit Q of $\{T^{2n}\}$ exists, and Q is an orthogonal projection.

Of course, if T is positive as well, then the above result applied to the square root of T implies that the whole sequence $\{T^n\}$ converges strongly, and so forth.

PROOF. Let x belong to H. Let $a_n = \|T^n x\|^2$. Then $\{a_n\}$ is bounded, since $|T| \leqq 1$, and is convex, since

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$$0 \leqq \|T^n x - T^{n+2}x\|^2 = \Delta^2 a_n \tag{2}$$

by the self-adjointness of T. Thus, (1) follows from (2), Lemma 1, and the fact that here $a_1 - \lim a_n \leqq a_1 \leqq \|x\|^2$. Since $\{T^{2n}x\}$ is a Cauchy sequence by

$$\lim_{m,n\to\infty} \|T^{2m}x - T^{2n}x\|^2 = \lim_{m,n\to\infty} (a_{2m} - 2a_{m+n} + a_{2n}) = 0,$$

the strong limit Q of $\{T^{2n}\}$ exists. From the self-adjointness of T, the self-adjointness and idempotence of Q easily follow. Therefore Q is an orthogonal projection.

LEMMA 3. *Suppose that $a_0, a_1, \cdots$ is a complex number sequence such that $c^2 = \sum_{n=1}^{\infty} n|a_n - a_{n+1}|^2 < \infty$. Let $b_n = \sum_{k=1}^{2^n} a_k/2^n$, $n = 0, 1, \cdots$. (i) Then $\sup_{1\leq n}|a_n| \leqq 3 \sup_{0\leq n}|b_n| + |c|$. (ii) If $\lim_{n\to\infty} b_n = a_0$ then $\lim_{n\to\infty} a_n = a_0$.*

Part (ii) is similar to a theorem of Fejér [4] who assumes slightly more, namely that $\lim_{n\to\infty} \sum_{k=1}^{n} a_k/n = a_0$.

PROOF. In the proof, we may and do assume that $a_0 = 0$. Let $c_n^2 = \sum_{k=2^n}^{\infty} k|a_k - a_{k+1}|^2$. Then

$$\max_{2^n \leqq j \leqq k \leqq 2^{n+1}} |a_j - a_k| \leqq \sum_{k=2^n}^{2^{n+1}-1} |a_k - a_{k+1}|$$
$$\leqq \left[2^n \sum_{k=2^n}^{2^{n+1}-1} |a_k - a_{k+1}|^2\right]^{1/2} \leqq |c_n|.$$

If m is a nonnegative integer and $2^m \leqq n \leqq 2^{m+1}$, then

$$|a_n| = \left| b_m - 2b_{m+1} + \sum_{k=2^m+1}^{2^{m+1}} (a_k - a_n)/2^m \right|$$
$$\leqq |b_m| + 2|b_{m+1}| + |c_m|.$$

Both (i) and (ii) easily follow.

In the following two theorems, let Q denote the strong limit of $\{T^{2n}\}$ in L_2, which necessarily exists by Lemma 2.

THEOREM 1. *Suppose that T is a linear self-adjoint operator with norm $|T| \leqq 1$ in $L_2(W, F, \mu)$ where (W, F, μ) is a positive measure space. (i) If x is in L_2 and*

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} T^{2k}x/2^n \tag{3}$$

exists almost everywhere, then

$$\lim_{n\to\infty} T^{2n}x = Qx$$

almost everywhere. (ii) *If x is in L_2 and $f(x) = \sup_{0\leq n} \left|\sum_{k=1}^{2^n} T^{2k}x/2^n\right|$ is in L_2, then $g(x) = \sup_{1\leq n} |T^{2n}x|$ is in L_2 and satisfies*

$$\|g(x)\| \leqq 3\|f(x)\| + \|x\|.$$

PROOF. By the monotone convergence theorem and by Lemma 2,

$$\int_W \sum_{n=1}^{\infty} 2n\,|T^{2n}x - T^{2n+2}x|^2 d\mu = \sum_{n=1}^{\infty} 2n\|T^{2n}x - T^{2n+2}x\|^2 \leqq \|x\|^2. \tag{4}$$

Thus, $\sum_{n=1}^{\infty} n\,|T^{2n}x - T^{2n+2}x|^2 < \infty$ almost everywhere.

Since $T^{2n} \to Q$ strongly, we have that $\sum_{k=1}^{2^n} T^{2k}/2^n \to Q$ strongly. Thus, if x satisfies the condition of (i), then the limit (3) must equal Qx almost everywhere, and, by Lemma 3, the conclusion of (i) follows.

Suppose that x satisfies the condition of (ii). Then, by (4) and Lemma 3, the conclusion of (ii) follows.

THEOREM 2. *Let (W, F, μ) be a positive measure space and T be a linear self-adjoint operator in $L_2(W, F, \mu)$ such that for each x in L_2,*

$$\int_W |Tx|\,d\mu \leqq \int_W |x|\,d\mu, \tag{5}$$

$$\operatorname*{ess\,sup}_W |Tx| \leqq \operatorname*{ess\,sup}_W |x|. \tag{6}$$

Then, for each x in L_2,

$$\lim_{n\to\infty} T^{2n}x = x$$

almost everywhere. Furthermore, for each x in L_2, $h(x) = \sup_{1\leq n} |T^n x|$ is in L_2 and satisfies

$$\|h(x)\| \leqq 21\|x\|. \tag{7}$$

Inequality (7) could be sharpened, perhaps considerably. The main point, of course, is that h maps the unit sphere into a bounded set. Also, note that since T is self-adjoint, if (5) is satisfied for all x in L_2, then so is (6), and conversely.

PROOF. The domain of T contains the μ-integrable simple functions. Therefore, by (5) and (6), and by the Riesz convexity theorem (more precisely, by its proof as given, for example, in [3, pp. 525–526]), if $1 \leqq p < \infty$, then T has a unique continuous extension to L_p with norm $\leqq 1$. (Since (6) is satisfied for x in $L_1 \cap L_\infty \subset L_2$, it is of no importance whether or not T can be extended to a linear operator in L_∞ with norm $\leqq 1$.) Thus, T as an operator in L_2 has norm $\leqq 1$ and therefore satisfies the conditions of Theorem 1. Two theorems of Dunford and Schwartz [2, Theorems 3.6 and 3.8] imply here, among other things, that the conditions of (i) and (ii) in Theorem 1 are satisfied for all x in L_2 and that the function f of Theorem 1 satisfies $\|f(x)\|^2 \leqq 8\|x\|^2$, x in L_2. Thus, g of that theorem satisfies

$$\|g(x)\| \leqq 10\|x\| \tag{8}$$

for x in L_2. From $0 \leqq h(x) \leqq g(x) + g(Tx) + |Tx|$ and (8) we see that h has the desired properties. This completes the proof.

3. **Applications to iterates of conditional expectation operators.** Let (W, F, P) be a probability space. If F_0 is a sub-σ-field of F, let $\overline{F}_0$ denote the smallest σ-field containing F_0 and the collection of all sets A in F satisfying $P(A)=0$. We recall (see [1, pp. 17–18]) that if F_0 is a sub-σ-field of F and x is an F-measurable (real or complex valued) function on W such that $\int_W |x| dP < \infty$, then the conditional expectation of x relative to F_0, written $E(x| F_0)$, is any $\overline{F}_0$-measurable function y satisfying $\int_A x dP = \int_A y dP$, A in F_0. (Alternatively, some authors require that $E(x| F_0)$ be F_0-measurable. The difference is of no importance here.) Throughout this section, let $F_1, F_2, \cdots$ be sub-σ-fields of F and let $T_1 = E(\cdot | F_1), \cdots$ be the corresponding conditional expectation operators in $L_1(W, F, P)$ where F-measurable functions equal almost everywhere are identified. It is well known, and easy to verify from the definition, that in L_2, T_k is an orthogonal projection with range $L_2(W, \overline{F}_k, P)$, and that in L_p, T_k has norm $\leqq 1$, $1 \leqq p \leqq \infty$. Let $S_n = T_n \cdots T_2 T_1$.

LEMMA 4. *If x is in L_2, then*

$$\lim_{n \to \infty} (S_n x - S_{n+1} x) = 0$$

almost everywhere and in L_2 norm, $\sup_{1 \leq n} |S_n x - S_{n+1} x|$ *is in L_2 and has norm* $\leqq \|x\|$.

PROOF. Let x be in L_2. Then for all n, $\|S_n x - S_{n+1} x\|^2 = \|S_n x\|^2 - \|S_{n+1} x\|^2$ using the self-adjointness and idempotence of T_{n+1}. This implies, with the aid of the monotone convergence theorem, that

$$\int_W \sum_{n=1}^{\infty} |S_n x - S_{n+1} x|^2 dP = \sum_{n=1}^{\infty} \|S_n x - S_{n+1} x\|^2 \leqq \|x\|^2. \tag{9}$$

Hence, $\sum_{n=1}^{\infty} |S_n x - S_{n+1} x|^2 < \infty$ almost everywhere and the first part of the lemma follows. The second part follows from (9) and

$$\sup_{1 \leqq n} \left| S_n x - S_{n+1} x \right| \leqq \left[\sum_{n=1}^{\infty} \left| S_n x - S_{n+1} x \right|^2 \right]^{1/2}.$$

THEOREM 3. *Suppose that $T_{2n-1} = T_1$ and $T_{2n} = T_2$ for each positive integer n. Then, for each x in L_2,*

$$\lim_{n \to \infty} S_n x = E(x \mid \overline{F}_1 \cap \overline{F}_2) \tag{10}$$

almost everywhere and in L_2 norm. Furthermore, if x is in L_2, then $f(x) = \sup_{1 \leqq n} \left| S_n x \right|$ is in L_2 and satisfies

$$\| f(x) \| \leqq 14 \| x \|.$$

PROOF. Let $T = T_1 T_2 T_1$. Then T satisfies the conditions of Theorem 2 and $T^{2n} = S_{4n+1}$ for $n \geqq 1$. Therefore, letting Q denote the strong limit of $\{T^{2n}\}$ in L_2, we have that if x is in L_2 then $\lim_{n \to \infty} S_{4n+1} x = Qx$ almost everywhere and $\sup_{1 \leqq n} \left| S_{4n+1} x \right|$ is in L_2 and, by (8), has norm $\leqq 10 \| x \|$. Thus, by Lemma 4, $\lim_{n \to \infty} S_n x = Qx$ almost everywhere and in L_2 norm, and $\sup_{1 \leqq n} \left| S_n x \right|$ is in L_2 with norm $\leqq 14 \| x \|$, x in L_2.

It remains to show that $Q = E(\cdot \mid \overline{F}_1 \cap \overline{F}_2)$ in L_2. By Lemma 2, Q is an orthogonal projection. Also, the restriction of $E(\cdot \mid \overline{F}_1 \cap \overline{F}_2)$ to L_2 is an orthogonal projection with range $L_2(W, \overline{F}_1 \cap \overline{F}_2, P) = \bigcap_{k=1}^{2} L_2(W, \overline{F}_k, P)$, the intersection of the ranges of the restrictions of T_1 and T_2 to L_2. Clearly, Q also has this set as its range, hence $Q = E(\cdot \mid \overline{F}_1 \cap \overline{F}_2)$ in L_2.

REMARKS. 1. The convergence in L_2 norm assertion of Theorem 3 can also be obtained as an immediate consequence of a theorem of von Neumann [**5**, p. 55].

2. If, in (10), the right hand side were replaced by $E(x \mid \overline{F_1 \cap F_2})$, the first statement of Theorem 3 would no longer be true. This is due to the fact that $\overline{F_1 \cap F_2}$ is not necessarily $\overline{F}_1 \cap \overline{F}_2$.

3. Results similar to those of Theorem 3 can be obtained for three or more conditional expectation operators. For example, if $T = T_1 T_2 T_3 T_2 T_1$ and $S_{8n+1} = T^{2n}$ for $n \geqq 1$, then

$$\lim_{n \to \infty} S_n x = E(x \mid \overline{F}_1 \cap \overline{F}_2 \cap \overline{F}_3)$$

almost everywhere and in L_2 norm for each x in L_2, and so forth.

4. Theorem 3 indicates that $\{S_n x\}$ converges almost everywhere for each x in L_2. With S_n defined as in Theorem 3, we have found a number of special but nontrivial cases in which $\{S_n x\}$ converges almost everywhere for each x in L_1 and no cases in which this is not true. We conjecture that none exist.

References

1. J. L. Doob, *Stochastic processes*, New York, Wiley, 1953.
2. Nelson Dunford and J. T. Schwartz, *Convergence almost everywhere of operator averages*, J. Rational Mech. Anal. vol. 5 (1956) pp. 129–178.
3. ———, *Linear operators*, part 1, New York, Interscience, 1958.
4. L. Fejér, *La convergence sur son cercle de convergence d'une série de puissance effectuant une représentation conforme du cercle sur le plan simple*, C. R. Acad. Sci. Paris vol. 156 (1913) pp. 46–49.
5. J. von Neumann, *Functional operators*, vol. 2, Princeton University Press, 1950.

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A THEOREM ON OVERCONVERGENCE

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The conjecture announced by A. J. Macintyre [2; 3] is equivalent to the theorem stated and proved below.

Theorem. *Let D be an open domain containing the origin and let $f(z)$ be a function regular in D with the expansion $f(z)=\sum_0^\infty c_nz^n$. Let D_1 be a bounded closed domain contained in D. Then there exists a positive number $\lambda_0=\lambda_0(D, D_1)$ such that if $c_n=0$ for a sequence of intervals $n_k\leqq n\leqq \lambda n_k$ with $\lambda>\lambda_0$, then the subsequence of partial sums $s_{n_k}=\sum_0^{n_k}c_nz^n$ converges uniformly to $f(z)$ in D_1.*

Proof. Let CD and CD_1 denote the complements of D and D_1 respectively and let h_i, $i=1, 2, \cdots$, be the components of CD_1. The components can be considered as disjoint and there exists only one unbounded component. The one unbounded component will be denoted as h_1.

One can assert that there exists only a finite number of components h_i such that

$$h_i\cap CD\neq\varnothing, \tag{1}$$

where $\varnothing$ is the empty set. This assertion is proved as follows. Assume that there exists an infinite number of components h_i, $i\geqq 2$, such that (1) is valid. A bounded sequence of points a_i can be formed where $a_i\in h_i\cap CD$, $i\geqq 2$. Every a_i is an element of CD and hence the dis-

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