LINE INVOLUTIONS IN S_3 WHOSE SINGULAR LINES ALL MEET A TWISTED CURVE

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Introduction. In §1 we investigate a new series of line involutions[1] in a projective space of three dimensions over the field of complex numbers. These are defined by a simple involutorial transformation of the points in which a general line meets a nonsingular quadric surface bearing a curve of symbol $(2, k-2)$. Then in §2 we show that any line involution with the properties that

(a) It has no complex of invariant lines, and
(b) Its singular lines[2] form a complex consisting exclusively of the lines which meet a twisted curve,

is necessarily of the type discussed in §1. No generalization of these results to spaces of more than three dimensions has so far been found possible.

1. Let Q be a nonsingular quadric surface bearing reguli R_1 and R_2, and let Γ be a $(2, k-2)$ curve of order k on Q. A general line l meets Q in two points, P_1 and P_2, through each of which passes a unique generator of the regulus, R_1, whose lines are simple secants of Γ. On these generators let P_1' and P_2' be, respectively, the harmonic conjugates of P_1 and P_2 with respect to the two points in which the corresponding generator meets Γ. The line $l' \equiv P_1' P_2'$ is the image of l. Clearly, the transformation is involutorial.

We observe first that no line, l, can meet its image except at one of its intersections with Q. For if it did, the plane of l and l' would contain two generators of R_1, which is impossible. Moreover, from the definitive transformation of intercepts on the generators of R_1, it is clear that the only points of Q at which a line can meets its image are the points of Γ. Hence the totality of singular lines is the kth order complex of lines which meet Γ.

The invariant lines are the lines of the congruence of secants of Γ, since each of these meets Q in two points which are invariant. The order of this congruence is $(k^2-5k+8)/2$, since $(a^2+b^2-a-b)/2$

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[1] References to other line involutions will be found in [1], and in the bibliographies in [2; 3].

[2] Following the terminology introduced in [4] we say that a line is singular for an involution if it intersects its image.

secants of a curve of symbol (a, b) on a quadric surface pass through an arbitrary point. The class of the congruence is $(k^2-k)/2$, since an arbitrary plane meets Γ in k points.

Since the complex of singular lines is of order k and since there is no complex of invariant lines, it follows from the formula [1]

$$m = 2i + k - 1 \tag{1}$$

that the order of the involution is $m=k-1$.

There are various sets of exceptional lines, or lines whose images are not unique. The most obvious of these is the quadratic complex of tangents to Q, each line of which is transformed into the entire pencil of lines tangent to Q at the image of the point of tangency of the given line. Thus pencils of tangents to Q are transformed into pencils of tangents. It is interesting that a 1:1 correspondence can be established between the lines of two such pencils, so that in a sense a unique image can actually be assigned to each tangent. For the lines of any plane, π, meeting Q in a conic C, are transformed into the congruence of secants of the curve C' into which C is transformed in the point involution on Q. In particular, tangents to C are transformed into tangents to C'. Moreover, if π_1 and π_2 are two planes intersecting in a line l, tangent to Q at a point P, the two free intersections of the image curves C_1' and C_2' must coincide at P', the image of P, and at this point C_1' and C_2' must have a common tangent l'. Hence, thought of as a line in a particular plane π, any tangent to Q has a unique image and moreover this image is the same for all planes through l.

Each generator, λ, of R_2 is also exceptional, for each is transformed into the entire congruence of secants of the curve into which that generator is transformed by the point involution on Q. This curve is of symbol $(1, k-2)$ since it meets λ, and hence every line of R_2, in the $(k-2)$ invariant points on λ and since it obviously meets every line of R_1 in a single point. The congruence of its secants is therefore of order $(k^2-5k+6)/2$ and class $(k^2-3k+2)/2$.

A final class of exceptional lines is identifiable from the following considerations: Since no two generators of R_2 can intersect, it follows that their image curves can have no free intersections. In other words, these curves have only fixed intersections common to them all. Now the only way in which all curves of the image family of R_2 can pass through a fixed point is to have a generator of R_1 which is not a secant but a tangent of Γ, for then any point on such a generator will be transformed into the point of tangency. Since two curves of symbol $(1, k-2)$ on Q intersect in $(2k-4)$ points, it follows that there are $(2k-4)$ lines of R_1 which are tangent to Γ. Clearly, any line, l, of any bundle having one of these points of tangency, T, as vertex will be transformed into the entire pencil having the image of the second intersection of l and Q as vertex and lying in the plane determined by the image point and the generator of R_1 which is tangent to Γ at T. A line through two of these points, T_1 and T_2, will be transformed into the entire bilinear congruence having the tangents to Γ at T_1 and T_2 as directrices.

A conic, C, being a (1, 1) curve on Q, meets the image of any line of R_2, which we have already found to be a $(1, k-2)$ curve on Q, in $k-1$ points. Hence its image, C', meets any line of R_2 in $k-1$ points. Moreover, C' obviously meets any line of R_1 in a single point. Hence C' is a $(1, k-1)$ curve on Q. Therefore, the congruence of its secants, that is the image of a general plane field of lines, is of order $(k^2-3k+2)/2$ and class $(k^2-k)/2$. Finally, the image of a general bundle of lines is a congruence whose order is the order of the congruence of invariant lines, namely $(k^2-5k+8)/2$ and whose class is the order of the image congruence of a general plane field of lines, namely $(k^2-3k+2)/2$.

2. The preceding observations make it clear that there exist line involutions of all orders greater than 1 with no complex of invariant lines and with a complex of singular lines consisting exclusively of the lines which meet a twisted curve Γ. We now shall show that any involution with these characteristics is necessarily of the type we have just described.

To do this we must first show that every line which meets Γ in a point P meets its image at P. To see this, consider a general pencil of lines containing a general secant of Γ. By (1), the image of this pencil is a ruled surface of order $(k-1)$ which is met by the plane of the pencil in a curve, C, of order $(k-1)$. On C there is a $(k-1):1$ correspondence in which the $(k-1)$ points cut from C by a general line, l, of the pencil correspond to the point of intersection of the image of l and the plane of the pencil. Since C is rational, this correspondence has k coincidences, each of which implies a line of the pencil which meets its image. However, since the pencil contains a secant of Γ it actually contains only $k-1$ singular lines. To avoid this contradiction it is necessary that C be composite, with the secant of Γ and a curve of order $k-2$ as components. Thus it follows that the secants of Γ are all invariant. But if this is the case, then an arbitrary pencil of lines having a point, P, of Γ as vertex is transformed into a ruled surface of order $k-1$ having $k-1$ generators concurrent at P. Since a ruled surface of order n with n concurrent generators is necessarily a cone, it follows finally that every line through a point, P, of Γ meets its image at P, as asserted.

Now consider the transformation of the lines of a bundle with vertex, P, on Γ which is effected by the involution as a whole. From the preceding remarks, it is clear that such a bundle is transformed into itself in an involutorial fashion. Moreover, in this involution there is a cone of invariant lines of order $(k-1)$, namely the cone of secants of Γ which pass through P. Hence it follows that the involution within the bundle must be a perspective de Jonquieres involution of order $(k-1)$ and the invariant locus must have a multiple line of multiplicity either $(k-2)$ or $(k-3)$. The first possibility requires that there be a line through P which meets Γ in $(k-1)$ points; the second requires that there be a line through P which meets Γ in $k-2$ points. In each case, lines of the bundles are transformed by involutions within the pencils they determine with the multiple secant. In the first case the fixed elements within each pencil are the multiple secant and the line joining the vertex, P, to the intersection of Γ and the plane of the pencil which does not lie on the multiple secant. In the second, the fixed elements are the lines which join the vertex, P, to the two intersections of Γ and the plane of the pencil which do not lie on the multiple secant. The multiple secants, of course, are exceptional and in each case are transformed into cones of order $(k-2)$.

Observations similar to these can be made at each point of Γ. Hence Γ must have either a regulus of $(k-1)$-fold secants or a regulus of $(k-2)$-fold secants. Moreover, if $k \geqq 3$, no two of the multiple secants can intersect. For if such were the case, either the plane of the two lines would meet Γ in more than k points or, alternatively, the order of the image regulus of the pencil determined by the two lines would be too high. But if no two lines of the regulus of multiple secants of Γ can intersect, then the regulus must be quadratic, or in other words, Γ must be either a $(1, k-1)$ or a $(2, k-2)$ curve on a nonsingular quadric surface.

We now observe that the case in which Γ is a $(1, k-1)$ curve on a quadric is impossible if the complex of singular lines consists exclusively of the lines which meet Γ. For any pencil in a plane containing a $(k-1)$-fold secant of Γ has an image regulus which meets the plane of the pencil in $(k-1)$ lines, namely the images of the lines of the pencil which pass through the intersection of Γ and the multiple secant, plus an additional component to account for the intersections of the images of the general lines of the pencil. However, if there is no additional complex of singular lines, the order of the image regulus of a pencil is precisely $(k-1)$. This contradicts the preceding observations, and so, under the assumption of this paper we must reject the possibility that Γ is a $(1, k-1)$ curve on a quadric surface.[3]

Continuing with the case in which Γ is a $(2, k-2)$ curve on a quadric Q, we first observe that the second regulus of Q consists precisely of the lines which join the two free intersections of Γ and the planes through any one of the multiple secants. For each of these lines meets Q in three points, namely two points on Γ and one point on one of the multiple secants.

Now consider an arbitrary line, l, meeting Q in two points, P_1 and P_2. If α is the multiple secant of Γ which passes through P_2 and β is the simple secant of Γ which passes through P_1, and if $A_1, A_2, \cdots, A_{k-2}$ are the points in which α meets Γ, and if P_1' is the image of P_1 on the generator β, it follows that the image of the line P_1A_i is $P_1'A_i$. Moreover, the image of P_1P_1' is this same line. Hence the pencil (P_1, α) is transformed into a ruled surface of order $(k-1)$ containing $(k-1)$ concurrent generators. This image surface must therefore be a cone; and hence every line of the pencil (P_1, α), including l, is transformed into a line through P_1', the harmonic conjugate of P, with respect to the two intersections of β and Γ. By an identical argument, with the roles of P_2 and P_1 interchanged, it follows that the other intercept of l is similarly transformed on the generator of R_1 which passes through P_2. This completes our proof.

Bibliography

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[3] This possibility leads to another series of involutions which we propose to discuss in a later paper.